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УНИВЕРСИТЕТ

Ю. А. Шварц

ECONOMIC AND MATHEMATICAL METHODS

ИЗДАТЕЛЬСКИЙ ЦЕНТР
"Perm University Press"

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ»

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Учебное пособие направлено на изучение области, находящейся на стыке экономики и математики, – использование математических методов для анализа различных экономических явлений и процессов. Описаны основные математические методы, используемые в экономике. Изложение теоретического материала сопровождается графическими иллюстрациями и примерами решения экономических задач с использованием компьютерных электронных таблиц.

Учебное пособие на английском языке предназначено для студентов, обучающихся по направлению подготовки бакалавров «Business Informatics», program «Information systems and Big data».

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1. APPLICATION OF MATHEMATICAL METHODS IN ECONOMICS

1.1. Mathematical Modeling of Economic Relationships

A variety of laws operate in the economics, there are relationships between economic indicators. For example, observations and studies show that there is a relationship between the price of good and quantity demanded (the law of demand), there is a relationship between the volume of production of the firm and its profits, etc. The study of cause-and-effect relationships is of the greatest practical interest.

Cause-and-effect relationship between two indicators exists when one indicator (result) changes its value as the value of another indicator (factor) changes.

Let us introduce notations:

y – resultant indicator (dependent variable),

x – indicator-factor (independent variable).

All economic indicators can be divided into resultant indicators and factors. This division is relative.

A cause-and-effect relationship can be a functional dependence (fig. 1). If there is only the one value of y for every x then this relationship is called **a functional dependence** and is described using the mathematical concept of function.

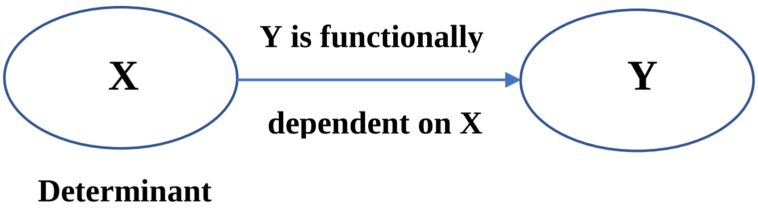

Fig. 1. Functional dependence between X and Y

The function or functional dependence is one of the main mathematical concepts, which are used to model the relationships between different quantities, including between economic indicators.

The functional dependence between x and y is denoted as follows:

$$y = f(x),$$

the value x is called an argument, and y is called a function. This notation means that y depends on x, or y is a function of x.

The following functions are used most frequently in economics:

1. The utility function is the dependence of consumer satisfaction on the quantities of goods consumed.

2. The production function expresses the relationship between the quantities of productive factors (such as labor and capital) used and the amount of product obtained.

3. The cost function is the dependence of production costs on output.

4. The supply function is the relationship between price and the quantity of a good that producers wish to sell.

5. The demand function is the dependence of the quantity demanded for a good on various factors, such as price of good, consumer income, etc.

The main task of economic analysis is to study the relationships of economic quantities, which are expressed in the form of functions.

In which direction would government revenue change if taxes were increased or import duties were imposed? Would a firm's revenue increase or decrease if the price of its product increased? In what proportion might additional equipment replace retiring workers? It is necessary to determine the functions of the relationship between economic quantities to solve such problems. These relationship functions are studied by the methods of differential calculus to answer these questions.

Analysis of the relationship of economic indicators should be carried out to obtain a function of the relationship. Four questions are answered sequentially:

1. **What are the factors** that determine the economic indicator?

2. **What is the sign** of this dependence?

3. **What is the measure** of this dependence?

4. **What is the functional expression** (mathematical equation) of this dependence?

Let us consider possible answers to these questions using the example of the simplest economic dependence, which is the demand function. The economic indicator under study will be the quantity demanded of a good. **The demand** is the quantity of good, that consumers want to buy at given prices and the consumer income.

1. What are the factors? It is necessary to list all the factors that determine the economic indicator under study in order to answer this question.

The quantity demanded is determined by the following factors (**demand factors**):

p – *price* of good,

I – *Income* of consumer,

p_c – price of goods that complement a given good in consumption (*the price of complementary goods*),

p_s – price of goods that are substitutes for a given good in consumption (*the price of substitute goods*),

p_e – *expected* future *price*,

I_e – *expected* future consumer *income*.

It can be written as follows:

$$q = f(p, I, p_c, p_s, p_e, I_e),$$

q is the quantity demanded of a good (*quantity*).

2. What is the sign of dependence? We need to determine directions in that result indicator (y) and factor (x) are moved.

A positive relationship occurs when y increases as x increases. **A negative (inverse) relationship** occurs when y decreases as x increases.

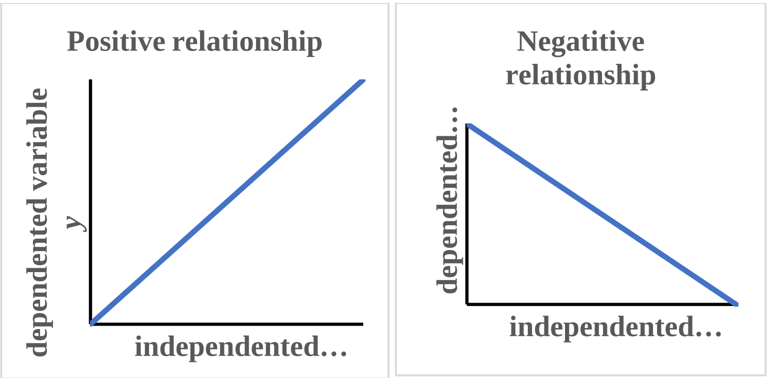

Fig. 2. Positive and negative relationships

Examples of a positive and negative relationship are shown in Fig. 2.

We know from the law of demand that quantity demanded decreases when the price of a good increases. There is a negative (inverse) relationship between the price of a good and quantity demanded. Another factor that determinates the demand for a good is consumer's income. If consumer's income increases then quantity of normal goods demanded increases and quantity of inferior goods demanded decreases. Consequently, the relationship between the consumer's income and the quantity of good demanded can be both positive and negative. The demand for a good also depends on the prices of related goods. These goods can be broken down into complements and substitutes. Quantity of a good demanded decreases as the price of complementary goods increases; quantity demanded increases as the price of substitute goods increases. When there is an expectation of rising prices or consumer income quantity demanded increases. Let's summarize the results in the tabl. 1:

Tabl. 1. Sign of dependence

Cause	Effect	Relationship
Price **increases**	quantity demanded **decreases**	**negative**
Income **increases**	quantity demanded **increases or decreases**	**positive or negative**
Price of complementary goods **increases**	quantity demanded **decreases**	**negative**
Price of substitute goods **increases**	quantity demanded **increases**	**positive**
Expected price **increases**	quantity demanded **increases**	**positive**
Expected income **increases**	quantity demanded **increases**	**positive**

Thus, if the factor and the resultant indicator move in the same direction, the relationship is positive; if they move in opposite directions, the relationship is negative.

3. What is the measure of dependence? To answer this question, we need to know how responsive the economic indicator is to a factor change. In other words, how much does indicator change when the factor changes? We need a measure of sensitivity. There are two approaches to analyzing the sensitivity of the dependence of y on x.

1) The incremental approach $(\Delta x \Rightarrow \Delta y)$. The absolute change Δx in the value of x is set and the corresponding change Δy in the value of y is analyzed (fig. 3).

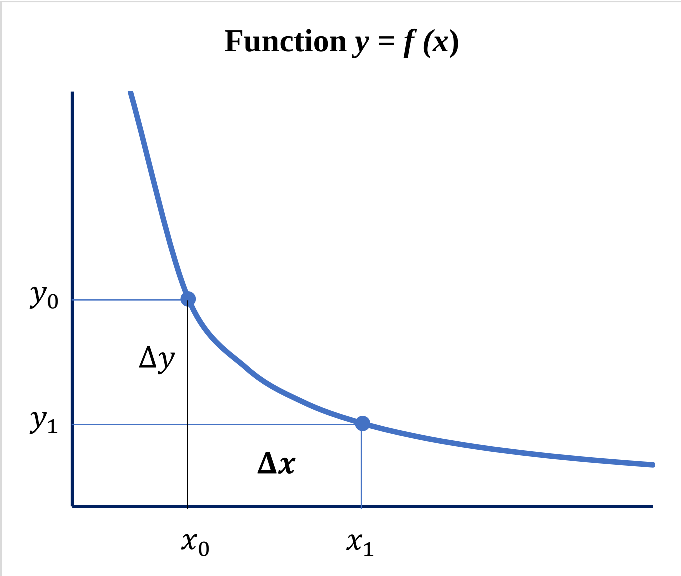

Fig. 3. Measure of functional dependence *y* on *x*

Absolute change of value *x* (increment *x*) is the difference between its new value and the original value:

$$\Delta x = x_1 - x_0.$$

Since the variable *y* is functionally dependent on *x*, and this dependence is described by the function, a change in the independent variable *x* by a value Δx will result in a change in the variable *y* by a value

$$\Delta y = f(x_1) - f(x_0) = y_1 - y_0 .$$

A measure of the sensitivity of *y* to a change in *x* is the ratio of the absolute changes in the variables *y* and *x*:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} .$$

Let's calculate the limit of the ratio of absolute changes

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x) ,$$

we obtain the derivative of the function $f(x)$, which shows the rate of change of the function.

One measure of the response of one variable *y* to changes in another variable *x* is the derivative

$$f^{'}(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \ ,$$

which characterizes the rate of change of the function when the argument changes. However, this indicator is inconvenient in economics because derivative depends on the choice of units.

For example, consider the demand function for sugar $q(p)$, where q is the quantity of good purchased by consumers at price p. Let the price of sugar be measured in rubles, if quantity of sugar demanded is measured in kilograms, then the derivative

$$q^{'}(p) = \frac{dq}{dp} = \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} \approx \frac{\Delta q}{\Delta p}$$

is measured in kg/rub. If quantity of sugar demanded is measured in quintals, then the derivative is measured in quintals/rub.

The value of the derivative will be different for different units of quantity demanded for the same price-demand relationship at the same price value. We have two different values to characterize the same relationship. This is the disadvantage of the derivative: its value depends on the choice of units. Therefore, the relationship not of absolute changes in the variables *x* and *y* (Δx and Δy), but of their relative or percentage changes is studied in economics to measure the sensitivity of the change in the function to a change in the argument.

2) **The tempo approach** ($\%\Delta x \Rightarrow \%\Delta y$). The percentage change in variable *x* is given and the corresponding percentage change in *y* is determined. The percentage change in a variable is its relative change expressed as a percentage.

The relative change in a variable is the ratio of the absolute change (Δx) in that variable to its original value (*x*), therefore the change is a fraction of the original value:

$$\frac{\Delta x}{x} = \frac{x_1 - x_0}{x_0},$$

The percentage change in the value of *x* is calculated by the formula

$$\%\Delta x = \frac{\Delta x}{x} \cdot 100\% = \frac{x_1 - x_0}{x_0} \cdot 100\% .$$

For example, let the price of bread increased from 20 to 30 rubles. Let's find the absolute, relative and percentage change in price. The computational formulas and results are shown in tabl. 2.

Tabl. 2. Absolute, relative and percentage change calculation

Variable	Absolute change	Relative change	Percentage change
x	$\Delta x = x_1 - x_0$	$\frac{\Delta x}{x} = \frac{x_1 - x_0}{x_0}$	$\%\Delta x = \frac{\Delta x}{x} \cdot 100\%$
P, price	$\Delta p = p_1 - p_0$	$\frac{\Delta p}{p} = \frac{p_1 - p_0}{p_0}$	$\%\Delta p = \frac{\Delta p}{p} \cdot 100\%$
Price	$\Delta p = 30 - 20 = 10 rub$	$\frac{\Delta p}{p} = \frac{10}{20} = 0.5$	$\%\Delta p = 0.5 \cdot 100\% = 50\%$

Calculate the ratio of percentage changes of the variables *x* and *y*

$$\frac{\%\Delta y}{\%\Delta x} = \frac{\frac{\Delta y}{y} \cdot 100\%}{\frac{\Delta x}{x} \cdot 100\%} = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} ,$$

we obtain that the ratio of percentage changes in *x* and *y* equals the ratio of their relative changes. Note also that this ratio is a dimensionless quantity.

The ratio of percentage (relative) changes in *x* and *y* is used in economics as a measure of the sensitivity of *y* to changes in *x*. This is a dimensionless quantity called the elasticity of a function (**the average elasticity of the function**):

$$\frac{\%\Delta y}{\%\Delta x} ,$$

the percentage change ratio limit

$$\lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x} \equiv \lim_{\Delta x \to 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} \equiv \frac{dy}{dx}\frac{x}{y} \equiv f'(x)\frac{x}{y}$$

is called the **elasticity of the function at the point**.

Elasticity is a dimensionless quantity that shows the ability of a function to respond to a change in its argument.

The elasticity of a function $y = f(x)$ (at a point) is the limit of the ratio of the percentage change in the dependent variable *y* to the percentage change in the independent variable *x* that caused the change. The elasticity of change of variable *y* with a change of variable *x* is denoted by $E_x(y)$, then we get the formula

$$E_x(y) = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x} = \frac{dy}{dx}\frac{x}{y} = y'\frac{x}{y}.$$

The notation $E_x(y)$ means the elasticity of *y* with respect to *x*. This elasticity is also called the marginal or point elasticity.

There is a term "coefficient of elasticity". The coefficient of elasticity is the value of the elasticity of a function calculated at a given point *x*. **The elasticity coefficient** shows the percentage change in the economic indicator under the influence of a single percentage change in the economic factor on which it depends, assuming that the values of the other factors affecting it are unchanged. Indeed,

$$E_x(y) = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x} \approx \frac{\%\Delta y}{\%\Delta x}.$$

If substitute $\%\Delta x = 1\%$, then we get from this formula: $E_x(y) \approx \%\Delta y$. So, the elasticity coefficient is an approximation of how much *y* will change when the factor *x* increases by 1%. For example, the elasticity coefficient is equal to 1: $E_x(y) = 1$, which means that when the factor *x* increases by 1%, the indicator *y* also increases by 1%. Let the elasticity coefficient is $E_x(y) = -2$, then if the factor *x* increases by 1%, the percentage change in *y* will be $\%\Delta y = -2\%$, so the indicator *y* will decrease by 2%. The fact that elasticity coefficient is approximate means that we should consider only small changes in independent variable *x*.

4. What is the functional expression of the dependence? It is necessary to specify the functional expression of the dependence under study (in the form of a equation) to answer this question. This equation can be obtained either from a theoretical model or from an econometric (empirical) study.

Let's return to the demand function for a good. Let all the factors influencing the quantity of good demanded be unchanged, except the price of good p, then we can consider the dependence of the quantity of good demanded on only one factor, the price. We obtain a one-factor demand function $q = f(p)$. Let's take a linear function (this is the simplest kind of relationship) to express the dependence of quantity demanded on price:

$$q = -ap + b, \quad a > 0,$$

slope coefficient is negative, because quantity demanded decreases when the price of a good increases.

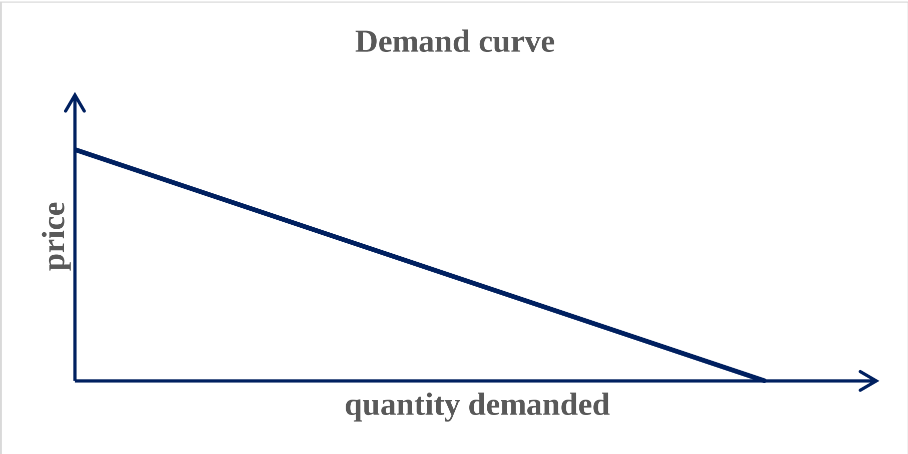

Fig. 4. Dependence of quantity demanded on price

The linear decreasing function is often used to model the relationship between the price of good and the quantity demanded. This relationship can be represented graphically. The graph of a decreasing linear function is a decreasing straight line. The graph of the dependence of demand on price is shown in fig. 4. This graph is called a demand cave. When graphing the demand curve, price goes on the vertical axes and quantity demanded goes on the horizontal axes. The demand curve illustrates the Law of demand.

1.2. Using Functional Dependence to Make Optimal Decisions

It is often necessary to find the best, or optimal, value of a particular indicator in the economy, such as maximum profit, maximum output, minimum cost, etc. Each indicator is a function of one or more arguments. For example, output can be viewed as a function of labor and capital costs (as is done in production functions), profit depends on the volume of production. Thus, finding the optimal value of an indicator is reduced to finding the extremum (maximum or minimum) of a function of one or several variables.

Let two economic indicators x and y be related by functional dependence, then determination of the optimal value of indicator y is the problem of finding the extremum of the function $y = f(x)$. A necessary condition for an extremum of a function of one variable is that its derivative is zero. We will consider this type of problems further.

Example 1. Find the maximal profit and the boundaries of the profitability of production for a given demand function $P(Q) = 150 - Q$ and cost function $C(Q) = 50 \cdot Q$, where Q is production volume, P is price per unit.

Analytical solution. Let's introduce the notation (tabl. 3):

Tabl. 3. Notation

Notation	Economic indicator
P	price per unit
Q	production volume
π	profit
R	revenue
C	costs

Profit π is the difference between the company's total revenue R from the sale of products and total costs C: $(Q) = R(Q) - C(Q)$, where $R(Q) = Q \cdot P(Q)$. For given functions of demand and costs the profit function is as follows:

$$\pi(Q) = Q \cdot P(Q) - C(Q) = -Q^2 + 100 \cdot Q.$$

1. Determine the volume of production Q^*, at which the maximal profit is achieved. Let's find the derivative of the profit function

$$\pi'(Q) = -2 \cdot Q + 100.$$

Equating the derivative to zero, we obtain the equation that has the solution $Q^* = 50$. This is a stationary point.

Find the second derivative of the profit function $\pi''(Q)$. If the second derivative is negative at a point Q^*, then Q^* is maximum point; if it is positive, then Q^* is minimum point. Since

$$\pi''(Q) = -2 < 0$$

at any volume of production, therefore, $Q^* = 50$ is maximum point.

Calculate the value of the profit function at the point $Q^* = 50$ to determine the maximal profit

$$\pi(50) = -50^2 + 100 \cdot 50 = -2500 + 5000 = 2500.$$

The maximal profit is 2500 monetary units.

2. Let's determine the boundaries of profitability of production. We find the break-even points from the equation $\pi(Q) = 0$:

$$-Q^2 + 100 \cdot Q = 0.$$

We take the common multiplier out of the parentheses and obtain the equation

$$-Q(100 - Q) = 0,$$

which has two roots $Q_1 = 0$, $Q_2 = 100$. These are break-even points.

The profit function $\pi(Q) = -Q^2 + 100 \cdot Q$ is a quadratic function, and its graph is a parabola, the branches of which are directed downward. The points of intersection of the parabola with the production volume axis Q are the break-even points. Profit has positive values at production volume $0 < Q < 100$.

Thus, the boundaries of profitability of production $0 < Q < 100$.

Graphical solution. Let's plot the revenue, cost and profit curves on the profitable production interval $0 < Q < 100$. Prepare the data for the graphs in spreadsheets (Fig. 5).

	A	B	C	D	E
1	**Q**	**P**	**C**	**R**	**Profit**
2	0	150	0	0	0
3	10	140	500	1400	900
4	20	130	1000	2600	1600
5	30	120	1500	3600	2100
6	40	110	2000	4400	2400
7	50	100	2500	5000	2500
8	60	90	3000	5400	2400
9	70	80	3500	5600	2100
10	80	70	4000	5600	1600
11	90	60	4500	5400	900
12	100	50	5000	5000	0
13	110	40	5500	4400	-1100

Fig. 5. Spreadsheet for graphing

Enter values of production volume Q from 0 to 110 in increments of 10 in column A. Enter the formula =150-A2, which is used to calculate the price, in cell B2. Enter the formula for calculating costs: =50*A2 in cell C2. The formula for calculating revenue: =A2*B2 is entered in cell D2, the formula for profit: =D2-C2 is entered in cell E2. Values in other cells are obtained by autofilling.

Select the range of cells A1:A13 to plot the revenue and cost curves, and then hold down CTRL and select the range C1:D13. Select the "Points" chart type. The graph of revenue and cost functions plotted using the data from these ranges is shown in Fig. 6.

The intersection points of the revenue and cost curves $Q = 0$ and $Q = 100$ are the break-even points. The revenue curve is above the cost curve on the interval $0 < Q < 100$, therefore production is profitable.

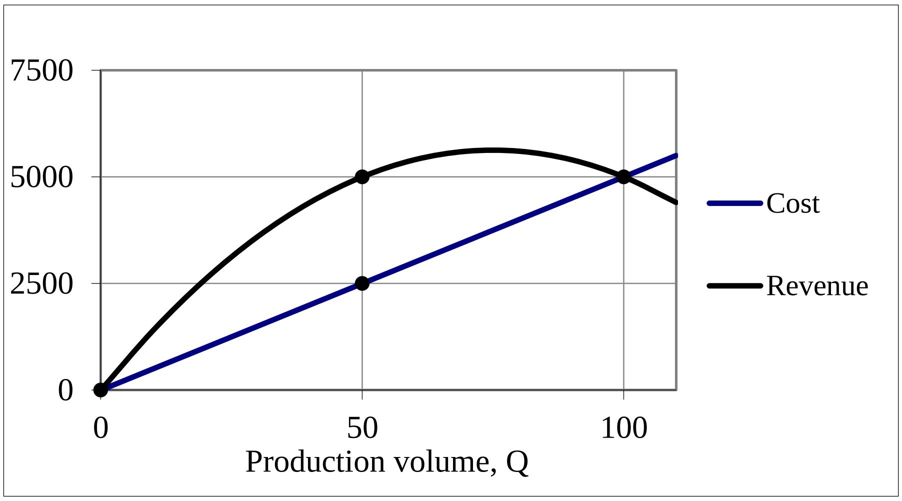

Fig. 6. Revenue and cost curves

The value of the distance between the revenue and cost curves at a certain value of production Q is equal to profit. The maximal distance is reached at the point $Q = 50$ where profit is maximal.

The graph of the profit function, plotted using data from the ranges of cells A1:A13 and E1:E13, is shown in Fig. 7.

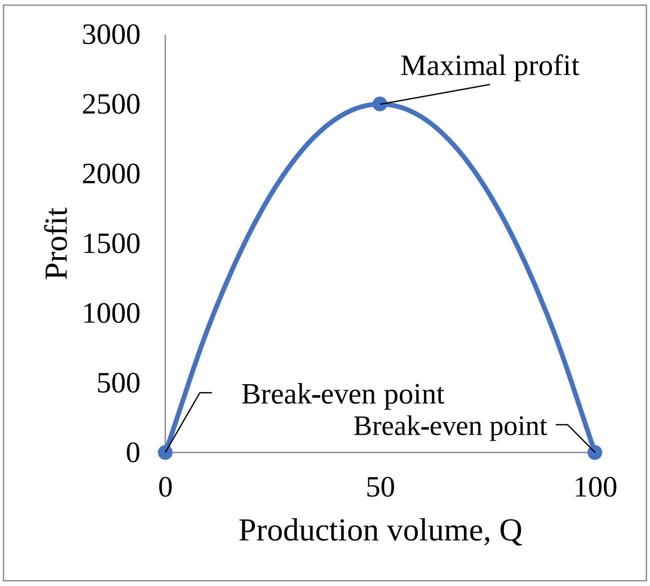

Fig. 7. Graph of the profit function

The points of intersection of the profit curve with the horizontal axis of production volume $Q = 0$ and $Q = 100$ are the break-even points. The profit curve is above the production volume axis on the interval $0 < Q < 100$, so production is profitable. The maximal profit value of 2500 is reached at $Q = 50$. The optimal production volume is 50 units.

Example 2. A producer sells his good at a price of 250 monetary units per unit, and the cost is a cubic dependence $C(Q) = 25Q + 3Q^3$, where Q is production volume. Find the optimal production volume for the producer and the corresponding profit.

Solution: Calculate the profit function as the difference between total revenue and costs:

$$\pi(Q) = R(Q) - C(Q) = 225Q - 3Q^3.$$

The derivative of the profit function is

$$\pi'(Q) = 225 - 9Q^2.$$

Let's find the stationary points from the equation

$$\pi'(Q) = 225 - 9Q^2 = 0,$$

where from $Q_1 = 5$, the second stationary point $Q_2 = -5$ is not considered in the sense of the problem.

Find the second derivative and determine its sign at $Q = 5$

$$\pi''(Q) = -18.$$

The second derivative is negative at any value of Q, including $Q = 5$. Therefore, profit is the maximal at $Q = 5$. Find the maximum of the profit function

$$max\,\pi\,(Q) = \pi(5) = 750.$$

The optimal production volume for the producer is 5 units and the corresponding profit is 750 monetary units.

Example 3. The profit function of the firm has the form: $\pi(Q) = Q^2 - 8Q + 10$. Find at which volume of production Q the profit will be maximal, if the capacity limit of the firm is Q_{max}.

Solution: It is necessary to find the maximum of the profit function $\pi(Q)$. Find the derivative of this function

$$\pi'(Q) = 2Q - 8.$$

Let's find the stationary points. Equate the derivative to zero and solve this equation with respect to Q, we obtain the stationary point $Q = 4$.

The derivative is negative at $Q < 4$ and positive at $Q > 4$, so $Q = 4$ is minimum point. Thus, the profit of the firm is minimal at the production volume of 4 units, this volume of production is not optimal for the firm (fig. 8).

What is the optimal production volume for the firm? To answer this question, we need to find the values of the profit function at the ends of the domain of function $0 \le Q \le Q_{max}$ and choose the maximum value from them.

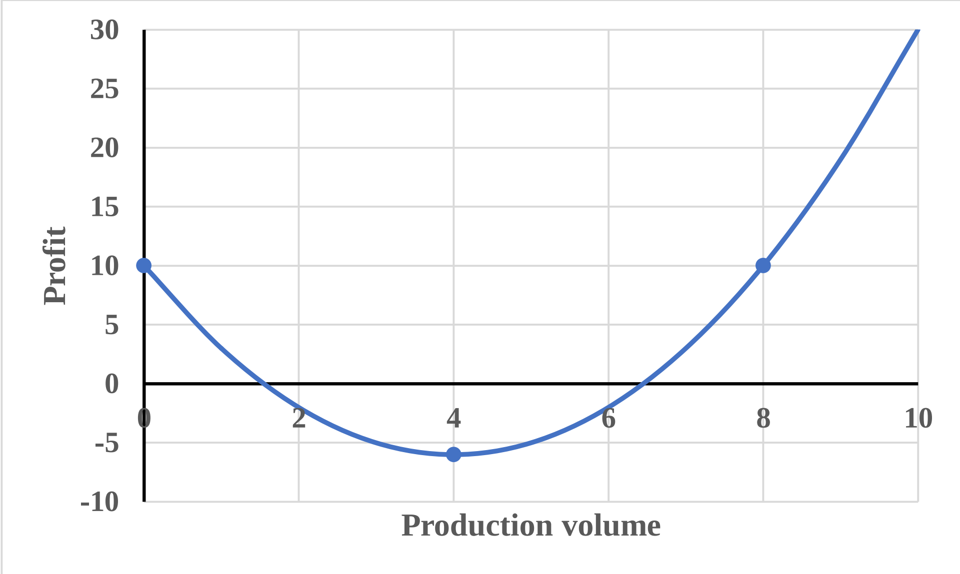

Fig. 8. Dependence of profit on volume production

Profit is equal to 10 units at the left end of the domain of function: $\pi(0) = 10$. In this case the firm does not produce anything, but receives revenue from renting the building and/or equipment. The same profit the firm would have with production volume equal to 8 units (fig. 8).

If the firm cannot produce more than 8 units ($Q_{max} \leq 8$), then the optimal solution for the firm is not to produce anything, but to receive revenue from renting the building and/or equipment, since the maximum value of the profit function is at the left end of the domain of function. If the firm is able to produce more than 8 units ($Q_{max} > 8$), then the optimal solution for the firm is to produce at the limit Q_{max} of its capacity, because the maximum value of the profit function is on the right end of the domain of function.

Thus, the optimal solution of this problem depends on the production capacity of the firm. This example shows that it is important to investigate the function to make optimal decisions.

Example 4. Four firms, whose supply volumes are known (in thousands of units):

$$Q_1 = 10; \;\; Q_2 = 8; \;\; Q_3 = 12; \;\; Q_4 = 5,$$

operate in the market of homogeneous products.

Market demand for a good is expressed by the function

$$P(Q) = 210 - 2Q,$$

where Q (in thousands of units) is the quantity of good demanded by the market at a price P.

The fifth firm, whose total production costs are given by the function

$$C = 150 + 80Q_5 + Q_5{}^2,$$

is going to enter this market. Here Q_5 (in thousands of units) is the quantity of good that the fifth firm produces.

What volume of production is profitable for the fifth firm to enter the market?

Solution: The goal of the fifth firm is to maximize profit

$$\pi(Q_5) = P(Q) \cdot Q_5 - C(Q_5),$$

where $Q = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 = 35 + Q_5$ is quantity demanded by the market at a price P. Then the demand function is

$$P(Q) = 210 - 2Q = 140 - 2Q_5,$$

the firm's profit is

$$\pi(Q_5) = -3Q_5{}^2 + 60Q_5 - 150.$$

The derivative of the profit function is

$$\pi'(Q) = -6Q_5 + 60.$$

The stationary point is $Q_5 = 10$. The second derivative is negative

$$\pi''(Q) = -6.$$

Therefore, production volume $Q_5 = 10$ provides the fifth firm with a maximum profit equal to

$$max\,\pi\,(Q) = \pi(10) = 150.$$

Answer: A firm needs to enter the market with a supply volume of 10 thousand units to make a maximum profit of 150 thousand monetary units.

1.3. Elasticity of a Function and its Properties

Elasticity is a measure of the functional relationship $y = f(x)$ between economic variables *x* and *y*. Let us consider the properties of elasticity, which simplify the calculation of elasticity. In deriving these properties, we will apply a formula that allows to calculate the elasticity of a function $y = f(x)$ using its derivative:

$$E_x(y) = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x} = \frac{dy}{dx}\frac{x}{y} = y'\frac{x}{y} \; . \qquad (1.1)$$

1. Elasticity is a dimensionless quantity, the value of which does not depend on the units in which the quantities *y* and *x* are measured:

$$E_{ax}(by) = E_x(y),$$

where *a* and *b* are constants. If the value of *y* needs to be converted to another unit of measurement, then *y* is multiplied by some constant. For example, let *y* be a value in kilograms, and you want to convert it to grams. It is necessary to multiply *y* by 1000, in order to get the value in grams. The constants *a* and *b* in the formula are constants that convert values *x* and *y* to other units.

Let's calculate the elasticity $E_x(by)$ to prove this property

$$E_{ax}(by) = \frac{d(by)}{d(ax)} \cdot \frac{ax}{by} = \frac{b \cdot dy}{a \cdot dx} \cdot \frac{ax}{by} = \frac{dy}{dx} \cdot \frac{x}{y} = E_x(y) \; .$$

In particular, $E_x(by) = E_x(y)$. Thus, if the function is multiplied by some constant, its elasticity does not change.

2. The elasticities of reciprocal functions are reciprocal values:

$$E_x(y) = \frac{1}{E_y(x)} \, .$$

Really,

$$E_x(y) = \frac{dy}{dx} \cdot \frac{x}{y} = \frac{1}{\frac{dx}{dy} \cdot \frac{y}{x}} = \frac{1}{E_y(x)} \, .$$

For example, the reciprocal functions are the dependence of quantity demanded on price of good and the dependence of price on quantity demanded. The elasticity of demand relative to price is the inverse of the elasticity of price relative to demand:

$$E_P(q) = \frac{1}{E_q(p)}.$$

3. The elasticity of the product of two functions $u(x)$ and $v(x)$, depending on the same argument x, is equal to the sum of elasticities:

$$E_x(uv) = E_x(u) + E_x(v).$$

Let us prove this property. Using formula (1.1), we get

$$E_x(uv) = (uv)' \cdot \frac{x}{uv} = \left(u'v + uv'\right) \cdot \frac{x}{uv} = u' \cdot \frac{x}{u} + v' \cdot \frac{x}{v} = E_x(u) + E_x(v).$$

Property 3 is proved.

4. The elasticity of the quotient of two functions $u(x)$ and $v(x)$, depending on the same argument x, is equal to the difference of elasticities:

$$E_x\left(\frac{u}{v}\right) = E_x(u) - E_x(v).$$

Let's calculate the elasticity of the quotient of two functions to prove it

$$E_x\left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)' \cdot \frac{x}{\frac{u}{v}} = \frac{u'v - uv'}{v^2} \cdot \frac{xv}{u} = u' \cdot \frac{x}{u} - v' \cdot \frac{x}{v} = E_x(u) - E_x(v).$$

5. The elasticity of the sum of two functions can be found by the formula

$$E_x(u+v) = \frac{uE_x(u) + vE_x(v)}{u+v}.$$

Let's get this formula. Let's calculate the elasticity of the sum of two functions

$$E_x(u+v) = (u+v)' \cdot \frac{x}{u+v} = (u' + v') \cdot \frac{x}{u+v} = \frac{u'x + v'x}{u+v} = \frac{uE_x(u) + vE_x(v)}{u+v}.$$

1.4. Computing the Elasticity

The elasticity of elementary function can be calculated by elasticity formula (1.1) using the derivative. The elasticities of some elementary functions are shown in tabl. 4.

Tabl. 4. Elasticities of elementary functions

Function	Equation	Derivative	Elasticity
Linear	$y = ax + b$	$y' = a$	$E_x(y) = \dfrac{ax}{ax+b}$
Power	$y = x^n$	$y' = nx^{n-1}$	$E_x(y) = n$
Exponential	$y = a^x$	$y' = a^x \cdot \ln a$	$E_x(y) = x \cdot \ln a$

The elasticity of a linear function is a variable value, and depends on at which point *x* it is calculated. Let define the elasticity of a linear decreasing function

$$y = ax + b,\ \ a < 0.$$

The domain of economic values of the variables *x* and *y* is given by the inequalities $x \geq 0$, $y \geq 0$. Let's introduce the notation $y_m = y(0) = b$, $x_m = -b/a = b/|a|$ which are maximum limits of variation of the variables *x* and *y*. The graph of the linear decreasing function is shown in fig. 9.

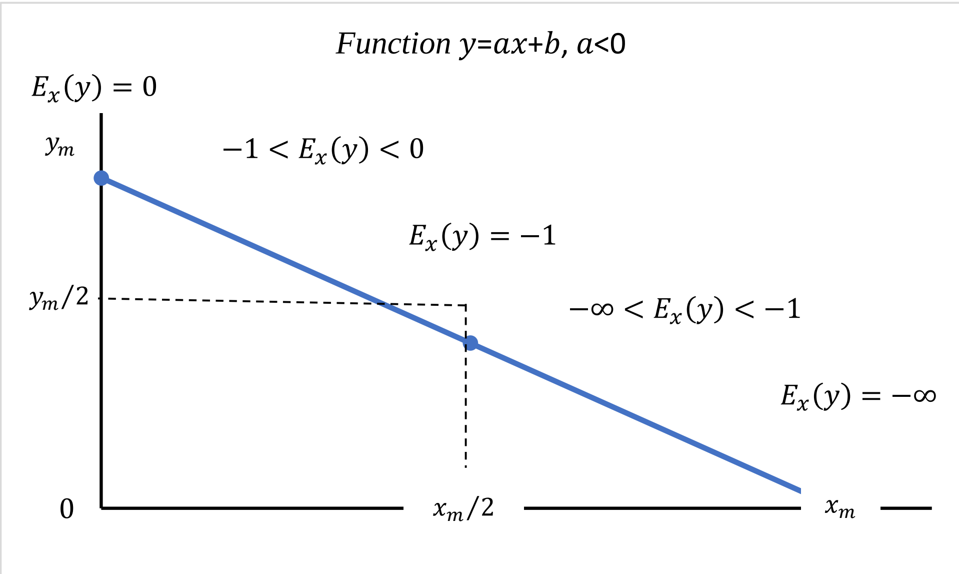

Fig. 9. Linear decreasing function

Let's calculate the elasticity of the linear decreasing function at the ends of the interval $0 \le x \le x_m$ and at the midpoint $x_m/2$ using the formula for the elasticity of a linear function from tabl. 4:

$$x = 0: \quad E_x(ax+b) = 0,$$

$$x = x_m: \ E_x(ax+b) = \frac{ax_m}{ax_m+b} = \frac{ax_m}{0} \to -\infty, \text{ since } a < 0,$$

$$x = x_m/2: \ E_x(ax+b) = \frac{ax_m/2}{ax_m/2+b} = \frac{-tg\alpha \cdot x_m/2}{y_m/2} = -\frac{y_m/2}{y_m/2} = -1.$$

Thus, the elasticity of a linear decreasing function varies from 0 at the point $\boldsymbol{x} = 0$ (point of intersection of the function graph with the *y*-axis) to the value $-\infty$ at the point $x = x_m$ (point of intersection of the function graph with the *x*-axis), passing through the value of -1 at the midpoint.

A function with infinite elasticity at all points is called **perfectly elastic**, a function with zero elasticity at all points is called **perfectly inelastic**.

Example 1. The dependence between the volume of production *x* (in million rubles) and the production cost *y* (in thousand rubles) is expressed by the function $y = -0{,}5x + 80$. Find the elasticity of production cost when production volume equals 60 million rubles.

Solving: The function $y = -0.5x + 80$ is a decreasing linear function. The graph of this function is shown in Fig. 10:

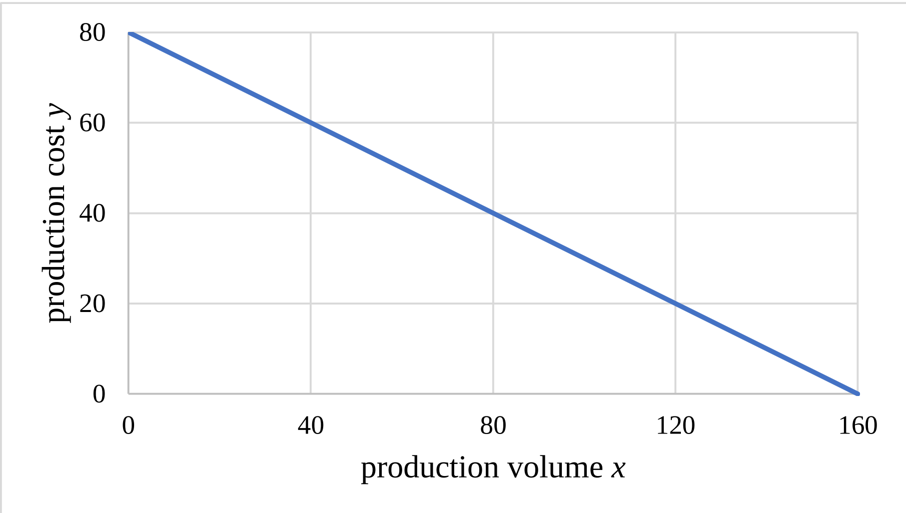

Fig. 10. Dependence of production costs on production volume

Let's determine the elasticity of cost using the formula for linear function elasticity (tabl. 4)

$$E_x(ax+b)=\frac{ax}{ax+b}.$$

In the given problem $= -0.5$, $b = 80$. Let's substitute these values into the linear function elasticity formula. We get

$$E_x(y) = \frac{-0.5x}{-0.5x + 80} = \frac{x}{x - 160}.$$

Let's calculate the value of the elasticity for the volume of production equal to 60 million rubles. We obtain

$$E_{x=60}(y) = -\frac{60}{100} = -0.6\,.$$

This means that an increase in volume of production by 1% will lead to a decrease in the cost of production by 0.6% as volume of production equal to 60 million rubles.

Let us consider examples of calculating elasticity of functions using elasticity properties from Section 1.3 and formulas from tabl. 4.

Example 2. Calculate the elasticity of a function

$$q = \frac{\alpha I(I + \beta)}{I^2 + \gamma},$$

where $\alpha = 10$, $\beta = 3$, $\gamma = 2$.

Solving: This function is Tornquist function that expresses the dependence of demand for inferior goods on consumer income. The role of the function *y* is played by demand *q*, the argument of the function is not *x*, but the income of consumers *I*. Using the above properties and Table 4, we change the notations in the formulas: *q* should be written instead of *y*, and *I* instead of *x*. Let us use property 4 (elasticity of the quotient of two functions). Then

$$E_I(q) = E_I\left(\frac{\alpha I(I + \beta)}{I^2 + \gamma}\right) = E_I\big(\alpha I(I + \beta)\big) - E_I(I^2 + \gamma)\,.$$

Let's apply property 3 (elasticity of product) to calculate the elasticity $E_I\big(\alpha I(I + \beta)\big)$

$$E_I\big(\alpha I(I + \beta)\big) = E_I(\alpha) + E_I(I) + E_I(I + \beta)\,.$$

The elasticity of the constant α is zero. The elasticity $E_I(I)$ of the power function is equal to the degree, that is one in this case. The elasticity of the linear function

$$E_I(I+\beta)=\frac{I}{I+\beta}.$$

The function $I^2+\gamma$ is a quadratic function. We calculate the elasticity of this function using elasticity formula (1.1), which relates the elasticity of the function to the derivative. Then

$$E_I(I^2+\gamma)=(I^2+\gamma)'\cdot\frac{I}{I^2+\gamma}=\frac{2I^2}{I^2+\gamma}.$$

We get

$$E_I\left(\frac{\alpha I(I+\beta)}{I^2+\gamma}\right)=1+\frac{I}{I+\beta}-\frac{2I^2}{I^2+\gamma}.$$

Let's reduce the expression to a common denominator:

$$E_I\left(\frac{\alpha I(I+\beta)}{I^2+\gamma}\right)=\frac{-\beta I^2+2\gamma I+\beta\gamma}{(I+\beta)(I^2+\gamma)}.$$

Substituting the values of α, β, γ, we obtain that the elasticity of demand for inferior goods is calculated by the formula

$$E_I(q)=\frac{-3I^2+4I+6}{(I+3)(I^2+2)}.$$

Example 3. The demand for a good is expressed by the Tornquist function for necessities

$$q=\frac{\alpha I}{I+\beta}.$$

Calculate the elasticity of this function with respect to income using property 4 (elasticity of the quotient of two functions) and formulas from table 4. We obtain

$$E_I(q)=E_I(\alpha I)-E_I(I+\beta)=1-\frac{I}{I+\beta}=\frac{\beta}{I+\beta}.$$

2. SUPPLY AND DEMAND MODELING

2.1. Application of Elasticity in Demand Analysis

The widespread use of the concept of "elasticity" in economics is caused by the prevalence in economic practice of the percentage (relative) method of assessing changes in indicators and comparing these changes. Elasticity serves as an important characteristic of stable regularities. Consider the simplest economic regularity. This is the demand function. The value of demand for any good is determined by many factors: the price of this good, the income of consumer, the prices of other goods (complementary and substitute goods), etc.

Demand elasticity measures how buyers respond to changes in one factor separately: price, income, etc., as long as the other factors do not change. For example, let the values of all factors affecting quantity demanded, except the price of good, do not change, then we obtain a one-factor demand function $q = f(p)$, describing the dependence of quantity demanded on price. The measure of the dependence of quantity demanded on price of good is the price elasticity of demand, which is used in the analysis and forecasts of pricing policy.

We will consider further the types of elasticity that are used in economics.

1. The price elasticity of demand for a good is approximately the percentage (relative) change in quantity demanded divided by percentage (relative) change in the price of that good:

$$E_p(q) = \lim_{\Delta p \to 0} \frac{\%\Delta q}{\%\Delta p} \approx \frac{\%\Delta q}{\%\Delta p} = \frac{\frac{\Delta q}{q_0}}{\frac{\Delta p}{p_0}}.$$

For example, let 5% decrease in price causes a 12% increase in quantity demanded then price elasticity of demand is approximately equal to $12\%/(-5\%) = -2.4$.

Price elasticity of demand measures the percentage change in quantity demanded for a good when the price of that good increases by one percent; it shows the responsiveness of consumers to a price change.

Let p_0 is initial price value, q_0 is corresponding value of quantity demanded (fig. 11), the price changes by Δp and becomes equal to p_1. The change in price causes the change Δq in quantity demanded, the quantity demanded becomes equal to q_1. The slope of the demand curve is determined by the absolute changes in price and demand

$$\delta = \frac{\Delta p}{\Delta q}.$$

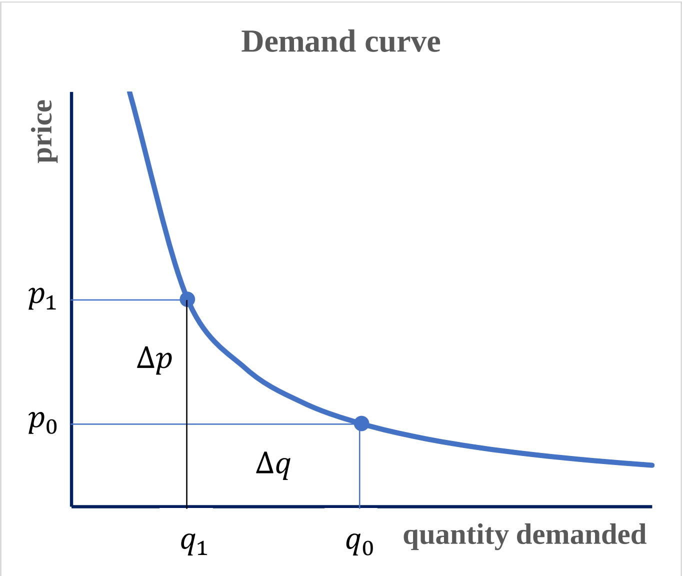

Fig. 11. Calculation of price elasticity of demand

Elasticity is the ratio of the relative changes in quantity demanded and price. It is equal to the inverse of the slope of the demand curve multiplied by the price and divided by the quantity of demand

$$E_p(q) \approx \frac{\frac{\Delta q}{q_0}}{\frac{\Delta p}{p_0}} = \frac{\Delta q}{\Delta p} \cdot \frac{p_0}{q_0} = \frac{1}{\delta} \cdot \frac{p_0}{q_0}.$$

The price elasticity of demand is negative $E_p(q) < 0$, because when the price increases, quantity demanded decreases, so percentage (relative) change in quantity demanded and percentage (relative) change in price have different signs. The absolute value of price elasticity of demand $|E_p(q)|$ is analyzed usually. Different categories of demand are distinguished depending on the absolute value of price elasticity of demand:

1. **Demand is** called **elastic** if the absolute value of the price elasticity of demand is greater than 1: $|E_p(q)| > 1$. If demand is a price elastic quantity demanded changes by lager percentage than price.

2. **Demand is** called **inelastic** if the absolute value of the price elasticity of demand is less than 1: $|E_p(q)| < 1$. If demand is a price inelastic a given percentage change in price results in a smaller percentage change in quantity demanded.

3. If the absolute value of price elasticity of demand is equal to 1: $|E_p(q)| = 1$, **demand is** called **unit elastic**. The absolute values of the percentage changes in quantity demanded and price are equal. How much the price increases, the same number of percent decreases in demand.

Let's draw a numerical axis and mark on it the values that the price elasticity of demand can have. Since the elasticity of demand is a negative, its values will be on the left of zero (fig. 12). The domain of values from -1 to 0 corresponds to inelastic demand, the domain of values from $-\infty$ to -1 is elastic demand, and the value equal to -1 is unit elastic demand.

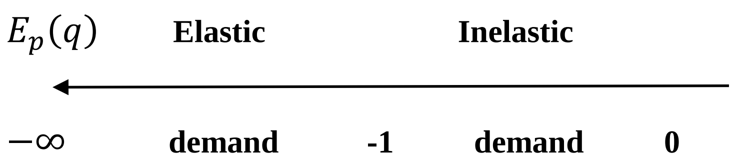

Fig. 12. Domain of Price elasticity of demand

If $E_p(q) = 0$, **demand is** called **perfectly inelastic**. In this case, the percentage change in quantity demanded is equal to zero. So, a change in price does not cause a change in demand. The demand curve is a vertical straight line (fig. 13).

If $E_p(q) = -\infty$, **demand is** called **perfectly elastic**. This means, that the percentage change in price approaches zero. Even a smallest price change leads to an enormous change in demand. The demand curve is a horizontal (fig. 13).

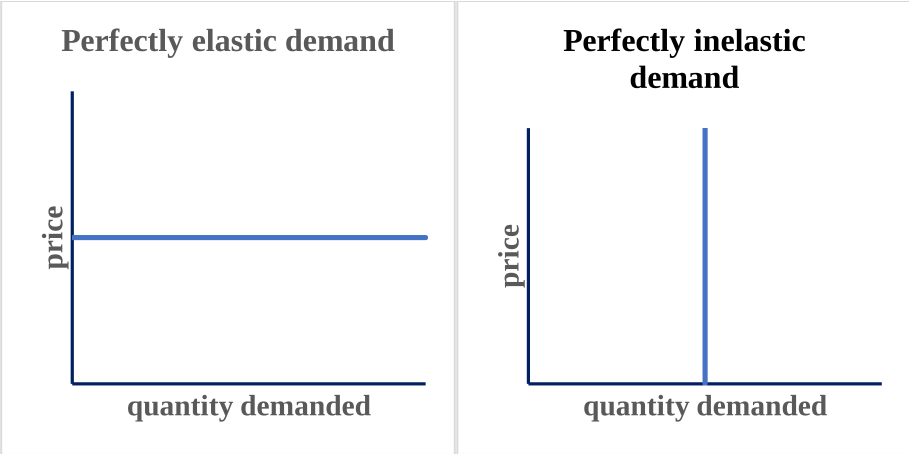

Fig. 13. Perfectly elastic and perfectly inelastic demand

On a linear demand curve, the price elasticity of demand has various values at different points. The value of the price elasticity of demand will rise from minus infinity to zero as we move down and to the right along the curve (fig. 14).

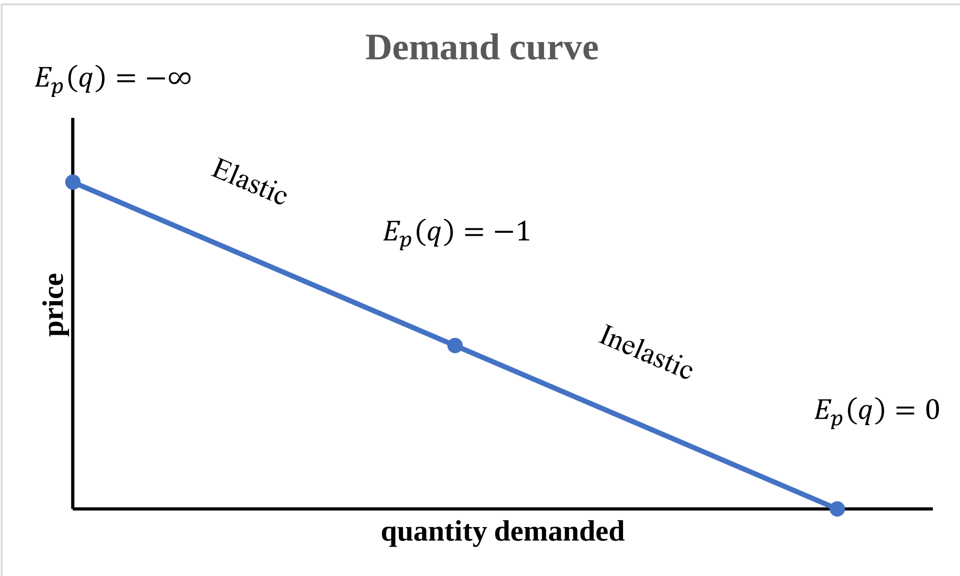

Fig. 14. Linear demand curve

Demand is elastic in the upper half of the linear demand curve and demand is inelastic in the lower half of the demand curve. Demand is unit elastic at the midpoint of the curve.

The price elasticity of demand for a good is related to the price elasticity of revenue.

2. Price elasticity of revenue. Consider the dependence of total revenue R on price. The price elasticity of revenue shows how much revenue changes when the price changes. The price elasticity of revenue is the limit of the ratio of the percentage change in total revenue to the percentage change in the price

$$E_p(R) = \lim_{\Delta p \to 0} \frac{\%\Delta R}{\%\Delta p} = \frac{dR}{dp}\frac{p}{R} = R'\frac{p}{R}.$$

Since total revenue is the price per unit times the number of units sold $R = p \cdot q(p)$, we get, using the elasticity property of the product of two functions

$$E_p(R) = E_p(p \cdot q) = E_p(p) + E_p(q) = 1 + E_p(q).$$

Taking into account that the price elasticity of demand is negative, this formula can be written in the form:

$$E_p(R) = 1 - \left|E_p(q)\right|.$$

Let's analyze this formula:

1. If $\left|E_p(q)\right| < 1$ (inelastic demand), then $E_p(R) > 0$. Thus, **the price elasticity of revenue is positive for inelastic demand**. What does it mean? Revenue increases when the price increases. What is the pricing policy in this case? If demand is inelastic, it is advantageous for sellers to raise the price so that revenue increases.

2. If $\left|E_p(q)\right| > 1$ (elastic demand), then $E_p(R) < 0$. **The price elasticity of revenue is negative when demand is elastic.** If demand is elastic, it is profitable for sellers to lower the price, then revenue increases. The revenue decreases, if the price goes up.

3. If $\left|E_p(q)\right| = 1$ (unitary elastic demand), then $E_p(R) = 0$. If there is a unitary elastic demand, then an increase or decrease in price does not cause a change in revenue.

Let's summarize the results in the tabl. 5:

Tabl. 5. Dependence of revenue on price for different categories of demand

	Price change	Change in total revenue	Change in quantity demanded
Price elastic demand $\lvert E_p(q) \rvert > 1$	Up	Down	Decrease by a larger percentage than price
	Down	Up	Increase by a larger percentage than price
Unit price elastic demand $\lvert E_p(q) \rvert = 1$	Up	Unchanged	Decrease by the same percentage as the price
	Down	Unchanged	Increase by the same percentage as the price
Price inelastic demand $\lvert E_p(q) \rvert < 1$	Up	Up	Decrease by a smaller percentage than price
	Down	Down	Increase by a smaller percentage than price

Thus, total revenue will move in the direction of the variable that changes by the larger percentage. If quantity demanded changes by a larger percentage than price (i.e., if demand is price elastic), total revenue will change in the direction of the quantity change. If price changes by a larger percentage than quantity demanded (i.e., if demand is price inelastic), total revenue will move in the direction of the price change. If price and quantity demanded change by the same percentage (i.e., if demand is unit price elastic), then total revenue does not change.

3. Income elasticity of demand measures the dependence of demand on income $q(I)$ assuming that other factors that influence demand are unchanged:

$$E_I(q) = \lim_{\Delta I \to 0} \frac{\%\Delta q}{\%\Delta I} = \frac{dq}{dI} \frac{I}{q}.$$

The income elasticity plays an important role in the analysis of changes in demand with small changes in income. The income elasticity of demand shows the percentage change in demand for a good when the income of consumers of those good increases by one percent. The income elasticity of demand varies in magnitude for different goods and may have negative values. Income elasticity of demand is positive for normal goods and negative for inferior goods. Thus, when consumers' income increases, the demand for normal goods increases, and the demand for inferior goods decreases. The dependencies of demand on income are shown in fig. 15.

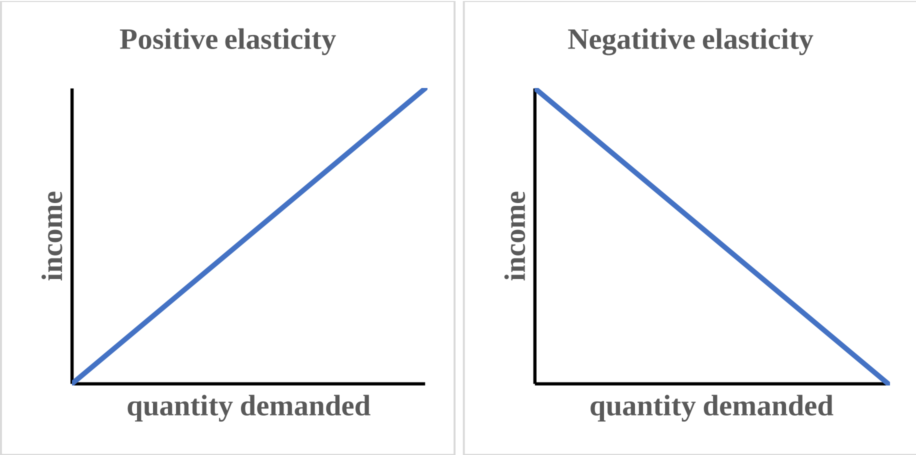

Fig. 15. Dependencies of demand on income

It is accepted to distinguish four groups of goods depending on the coefficient of income elasticity of demand:

1. inferior goods: $E_I(q) < 0$;
2. goods with low elasticity: $0 < E_I(q) < 1$;
3. goods with medium elasticity: $E_I(q)$ is approximately equal to 1;
4. goods with high elasticity: $E_I(q) > 1$.

Such goods as bread and inferior goods are low-value goods, these are goods with negative income elasticity of demand. According to the results of the surveys, the elasticity coefficients for basic foodstuffs are in the range from 0.4 to 0.8; for clothing, fabrics, shoes – in the range from 1.1 to 1.3. As income increases, demand shifts from goods of the first and second groups to goods of the third and fourth groups, while consumption of goods of the first group decreases in absolute value.

4. Cross price elasticity of demand characterizes the dependence of demand for a good on the price of another good (complementary or substitute goods).

Consider the effect on demand for a good of a change in the price of other good. The coefficient that shows by how much demand for a given good will change if the price of another good changes by 1%, under the condition that other prices and incomes of customers are unchanged, is called the cross-elasticity coefficient.

Let q_A is demand for good A, p_B is price of good B. Cross price elasticity of demand is approximately the percentage change in quantity demanded of good A divided by percentage change in the price of good B:

$$E_{p_B}(q_A) = \lim_{\Delta p_B \to 0} \frac{\%\Delta q_A}{\%\Delta p_B} \approx \frac{\%\Delta q_A}{\%\Delta p_B} = \frac{\frac{\Delta q_A}{q_A}}{\frac{\Delta p_B}{p_B}} = \frac{\Delta q_A}{\Delta p_A} \cdot \frac{p_B}{q_A}.$$

Goods can be divided into substitute goods and complementary goods according to the sign of the cross-elasticity coefficients (fig. 16). If $E_{p_B}(q_A) > 0$, this means that product *A* substitutes for product *B* in consumption, so that demand switches to product *A* when the price of product *B* increases. Examples of substitute goods are many food-stuffs.

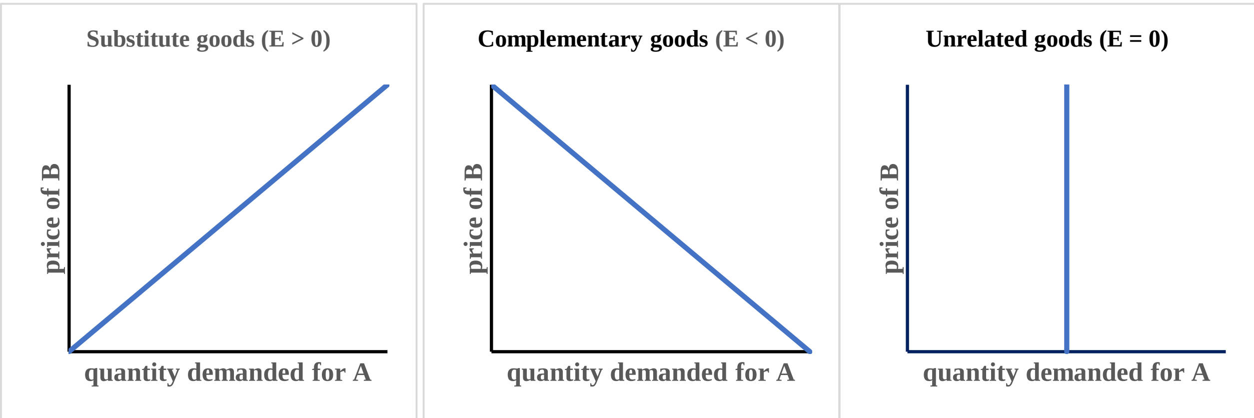

Fig. 16. Different values of cross elasticity of demand

If $E_{p_B}(q_A) < 0$, this means that product *A* complements product *B* in consumption, so that an increase in the price of product *B* leads to a decrease in demand for product *A*. An example of complementary goods are cars and gasoline.

The values of direct and cross coefficients of price elasticity of demand for one of the categories of families are shown in tabl. 6 [1]. The row is the demand for goods, and the columns are the prices of goods. Analyzing the values of direct coefficients of price elasticity of demand (on the diagonal) we can conclude that

- The demand for food is a price inelastic: $|E_p(q)| < 1$.
- The demand for clothing, fabrics and shoes is a unit price elastic: $|E_p(q)| = 1$.
- The last two groups of goods (furniture, housewares and cultural goods) are goods with a high price elasticity of demand: $|E_p(q)| > 1$.

The cross-elasticity coefficients in the "Foodstuffs" line are positive, it means that an increase in prices for manufactured goods increases demand for food (a decrease in demand for manufactured goods will free up funds for food).

Tabl. 6. Direct and cross elasticity coefficients

	Product Groups	Price			
		Foodstuffs	Clothing, fabrics, shoes	Furniture, housewares	Cultural Goods
Demand	Foodstuffs	-0.7296	0.0012	0.0043	0.0045
	Clothing, fabrics, shoes	-0.1991	-1.0000	0.0071	0.0074
	Furniture, housewares	-0.2458	0.0024	-1.2368	0.0092
	Cultural Goods	-0.2494	0.0024	0.0089	-1.2542

The cross-elasticity coefficients in the "Foodstuffs" column are negative. This means that when food prices increase, the demand for manufactured goods decreases (the increase in food prices reduces the amount of funds for the purchase of other goods).

Analyzing the values of the off-diagonal elements of this table, we can conclude that all industrial goods (the second, third, fourth columns and the second, third, fourth rows) are substitute goods: the cross-elasticity coefficients are positive.

2.2. Price Elasticity of Demand and Changes in Total Revenue

We know from the previous section that an increase in price leads to an increase in total revenue when demand is inelastic and a decrease in total revenue when demand is elastic. If demand is unit elastic, total revenue does not change when the price changes.

A demand curve can also be used to show changes in total revenue. Total revenue is given by the area of a rectangle drawn with point on the curve in the upper right-hand corner and the origin in the lower left-hand corner (fig. 17). The height of the rectangle is price; its width is quantity demanded. We can see from the figure that the area of the rectangle (total revenue) changes as we move along the demand curve.

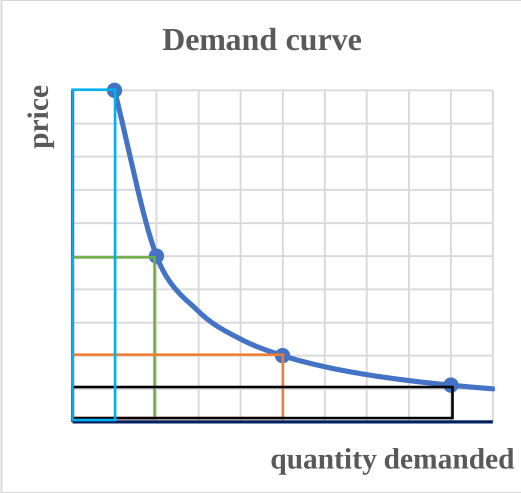

Fig. 17. Demand curve and total revenue

When a firm sets the price of a good, it wants to know how demand and total revenue will change. The demand curve shows how much of a good can be sold in a market at a given price. We can also determine the total revenue using the demand curve. The choice of price depends on what the firm wants to achieve with its price policy. Here various options are possible: increase in sales volume, maximum profit, etc. Consider solving such problems.

Example 1. Let the analysis of customer demand show that the demand function has the form:

$$q = 7 - p\,.$$

The demand curve is a decreasing straight line (fig. 18). Let's find at what price the revenue will be maximal. Total revenue is the price multiplied by the quantity demanded

$$R(p) = p \cdot D(p) = p \cdot (7 - p) = 7p - p^2\,.$$

Consequently, total revenue is a quadratic function, and the graph of total revenue is the parabola shown in fig. 18. The derivative of total revenue

$$R'(p) = 7 - 2p$$

is zero at the point $p = 3.5$. The derivative is positive before the point $p = 3.5$ and negative after the point. So, this point is the maximum point and revenue is maximal at the price of 3.5 monetary units.

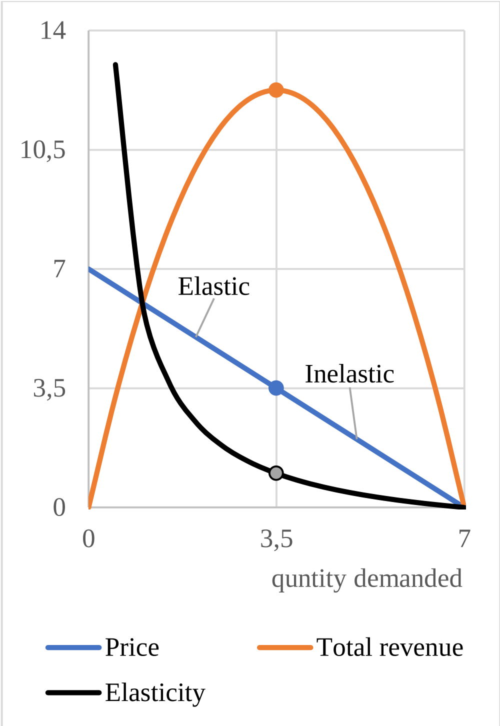

Fig. 18. Demand cave, total revenue and absolute value of elasticity

The price elasticity of demand in the case of demand curve $q = 7 - p$ is calculated by the formula

$$E_p(q) = -\frac{p}{7-p}.$$

From this formula we see that when the price changes from 0 to 7 the elasticity decreases from 0 to $-\infty$, and equals -1 at $p = 3.5$:

$$E_{p=0}(q) = 0, \quad E_{p=3.5}(q) = -1, \quad E_{p=7}(q) = -\infty.$$

The graph of the absolute value of price elasticity of demand (fig. 18) shows that demand is inelastic before price 3.5, elastic after price 3.5, and unit elastic at price 3.5. An increase in price means an increase in risk for the firm, because the quantity of good demanded decreases. The price elasticity of demand graph shows by how much the quantity demanded decreases when prices rise, and by how much it can increase when prices fall. An increase in price when demand is inelastic leads to a smaller decrease in demand and an increase in total revenue. If demand is elastic, an increase in price will result in a large decrease in demand and a decrease in total revenue. The fig. 18 shows this relationship between the change in total revenue and the price elasticity of demand.

The calculations on the Microsoft Excel sheet were used to plot the price, total revenue, and elasticity:

	A	B	C	D
1	***q***	***p***	***R***	***E***
2	0	7	0	
3	0,5	6,5	3,25	13
4	1	6	6	6
5	1,5	5,5	8,25	3,67
6	2	5	10	2,5
7	2,5	4,5	11,25	1,8
8	3	4	12	1,33
9	3,5	3,5	12,25	1
10	4	3	12	0,75
11	4,5	2,5	11,25	0,56
12	5	2	10	0,4
13	5,5	1,5	8,25	0,27
14	6	1	6	0,17
15	6,5	0,5	3,25	0,08
16	7	0	0	0

Fig. 19. Data for plotting the graphs

The values of quantity demanded from 0 to 7 with step 0.5 are entered in column A. The price is calculated in cell B2 by the formula =7-A2. Cell C2 contains the formula for calculating the total revenue: =A2*B2. The absolute value of the elasticity is calculated in cell D3 by the formula: =ABS((A3-7)/A3). The values in the other cells of the columns are obtained by autofilling. The range of cells A2:D16 is used to plot the graph shown in the fig. 18.

The price elasticity of demand depends on the nature of demand curve itself. Consider another demand curve in the next example.

Example 2. The demand function for a good has the form $p(q) = -q^2 - 6q + 135$, where q is the quantity of the good bought at price p per unit.

Let's find the price elasticity of demand. Since the demand function is given as the dependence of the price of good on their quantity, we will use the property of elasticity for reciprocal functions:

$$E_P(q) = \frac{1}{E_q(P)}.$$

Let's find the elasticity of price with respect to the quantity of good $E_q(p)$ using a formula relating elasticity to the derivative

$$E_q(p) = \frac{dP}{dq} \cdot \frac{q}{P},$$

where derivative of price with respect to quantity q is used. Then

$$E_q(p) = \frac{d}{dq}(-q^2 - 6q + 135) \cdot \frac{q}{-q^2 - 6q + 135} = \frac{-2q^2 - 6q}{-q^2 - 6q + 135}.$$

Multiply the numerator and denominator of the fraction by -1, and finally we get

$$E_q(p) = \frac{2q^2 + 6q}{q^2 + 6q - 135}.$$

So, the price elasticity of demand is calculated by the formula

$$E_P(q) = \frac{q^2 + 6q - 135}{2q^2 + 6q}.$$

Let find the quantity demanded at which demand becomes inelastic from equation $|E_P(q)| = 1$. Since $E_P(q) < 0$, we solve the equivalent equation

$$E_P(q) = -1,$$

substituting the found expression for the price elasticity of demand, we obtain

$$\frac{q^2 + 6q - 135}{2q^2 + 6q} = -1 .$$

From where we get

$$q^2 + 6q - 135 = -2q^2 - 6q.$$

If we reduce similar terms, we get an equation:

$$-3q^2 - 12q + 135 = 0.$$

Divide both sides of the equation by -3, and finally we have the following quadratic equation

$$q^2 + 4q - 45 = 0.$$

The equation has two roots: $q_1 = 5,\ q_2 = -9$. One root $q = 5$ satisfies the economic sense (quantity of goods $q \geq 0$). Thus, demand becomes inelastic when the quantity demanded is 5 units of a good. We find the corresponding price by substituting the value $q = 5$ into the demand function $p(q) = -q^2 - 6q + 135$. We obtain $p(5) = 80$.

Let's plot the price elasticity of demand in Microsoft Excel in the domain of economic values: $q \geq 0, p \geq 0$. Let's find the maximum possible value of the quantity q of good from the equation $p(q) = 0$:

$$q^2 + 6q - 135 = 0.$$

This equation has two roots: $q_1 = 9$, $q_2 = -15$. Only the first root $q = 9$ satisfies the economic sense (quantity of goods $q \geq 0$). Thus, the domain of economic values of quantity demanded: $0 \leq q \leq 9$. Let's eliminate the value $q = 0$ when we calculate the points for plotting, because the elasticity $E_P(q)$ at this point is not defined (the denominator is 0).

Let's prepare a table (Fig. 20) in Microsoft Excel to plot the elasticity of demand $E_P(q)$. Enter the demand values q from 1 to 9 in increments of 1 in column A. Calculate the modulus of price elasticity of demand $|E|$ in column B. We use the formula =ABS((A2^2+6*A2-135)/(2*A2^2+6*A2)). The graph of the price elasticity of demand, based on the range of cells A1:B10, is shown in Fig. 21.

	A	B
1	Q	|E|
2	1	16
3	2	5,95
4	3	3
5	4	1,70
6	5	1
7	6	0,58
8	7	0,31
9	8	0,13
10	9	0

Fig 20. Table for plotting the graph

The demand is elastic, when the quantity demand q does not exceed 5 units. Demand is unit elastic at the point $q = 5$. If $5 < q < 9$ demand is inelastic.

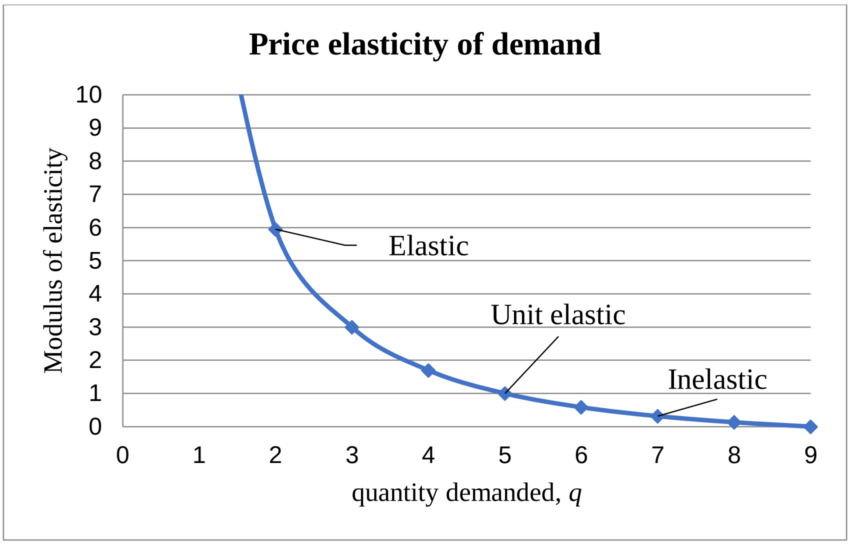

Fig. 21. The graph of the price elasticity of demand

The following examples show how a firm can calculate and use the price elasticity of demand to increase total revenue.

Example 3. A flower seller increased the price from 50 to 55 rubles per piece. The price increase caused a decrease in the quantity demanded from 150 to 120 pieces per day. Is the demand for flowers elastic? Is the price increase justified economically? Will the seller's revenue increase? Find the elasticity of revenue, the percentage change in revenue. How does the seller need to change the price to increase revenue by 10%?

Solving:

Percentage change in price

$$\%\Delta p = \frac{\Delta p}{p} \cdot 100\% = \frac{p_1 - p_0}{p_0} \cdot 100\% = \frac{55 - 50}{50} \cdot 100\% = 10\%.$$

Percentage change in quantity demanded

$$\%\Delta q = \frac{\Delta q}{q} \cdot 100\% = \frac{q_1 - q_0}{q_0} \cdot 100\% = \frac{120 - 150}{150} \cdot 100\% = -20\%.$$

Price elasticity of demand

$$E_p(q) \approx \frac{\%\Delta q}{\%\Delta p} = \frac{-20\%}{10\%} = -2.$$

The demand for flowers is elastic because $|E_p(q)| > 1$.

Initial revenue

$$R_0 = p_0 q_0 = 50 \cdot 150 = 7500 \text{ rubles}.$$

New revenue3

$$R_1 = p_1 q_1 = 55 \cdot 120 = 6600 \text{ rubles.}$$

The price increase caused a decrease in revenue.

Percentage change in revenue

$$\%\Delta R = \frac{\Delta R}{R} \cdot 100\% = \frac{R_1 - R_0}{R_0} \cdot 100\% = \frac{6600 - 7500}{7500} \cdot 100\% = -12\%.$$

Revenue decreased by 12%.

Price elasticity of revenue

$$E_p(R) = 1 - \left|E_p(q)\right| = 1 - |-2| = -1.$$

Since the price elasticity of revenue is approximately the percentage change in revenue divided by the percentage change in price:

$$E_p(R) \approx \frac{\%\Delta R}{\%\Delta p},$$

then percentage change in price

$$\%\Delta p \approx \frac{\%\Delta R}{E_p(R)} = \frac{10\%}{-1} = -10\%.$$

The price should be reduced by 10% then the revenue will increase by 10%. Demand for flowers is price elastic, so revenue moves in the direction opposite to the change in price. Revenue falls after a price increase. Revenue increases when the price goes down.

2.3. Supply and Demand Functions

Demand is the quantity of a good that consumers purchase in the marketplace per unit of time. Demand for a good depends on a number of factors. The simplest model of demand considers the dependence of demand on only one factor – the price of good. In this case the demand function has the form: $q = f(p)$. The reciprocal function $p = g(q)$ is considered often. Both functions are called demand functions, these functions are decreasing.

Supply is the quantity of good that producers offer for sale per unit of time. The dependence of supply on price is described by a supply function of the form $s = f(p)$. The reciprocal function of supply is as follows: $p = g(s)$. Both functions are increasing.

The point where the supply and demand curves intersect is called the **market equilibrium point**. The price and quantity corresponding to this point are called **the equilibrium price and the equilibrium sales volume**.

Example. Find the equilibrium price and the equilibrium sales volume if the demand function

$$D(q) = -5q + 150,$$

where q is the quantity of good purchased in the market per unit time, and the supply function

$$S(q) = \frac{q^2}{4} + \frac{q}{2} + 70,$$

where q is the quantity of good offered for sale per unit of time.

Analytic solution of the problem. Let's find the quantity of good at which the equilibrium price is reached. We have to solve the equation

$$D(q) = S(q).$$

Substitute the supply and demand functions from the problem condition into this equation

$$-5q + 150 = \frac{q^2}{4} + \frac{q}{2} + 70.$$

Let's multiply the left and right sides of the equation by 4, so that there are no fractional coefficients in the equation

$$-20q + 600 = q^2 + 2q + 280.$$

Let's reduce similar terms in the equation. We obtain a quadratic equation

$$q^2 + 22q - 320 = 0.$$

Find the discriminant to solve the quadratic equation

$$\frac{D}{4} = \left(\frac{b}{2}\right)^2 - ac = 11^2 + 320 = 441.$$

Find the roots of the equation by the formula

$$q_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\frac{D}{4}}}{a} = -11 \pm 21.$$

The equation has two roots

$$q_1 = 10,\ \ q_2 = -32.$$

One root $q = 10$ satisfies the economic sense (quantity of goods $q \geq 0$). Thus, the equilibrium sales volume is 10 units of goods.

Find the equilibrium price. Let's substitute the value $q = 10$ into the demand function $p = D(q) = -5q + 150$, we get $p = D(10) = 100$

Thus, the equilibrium price is 100 monetary units; the equilibrium sales volume is 10 units.

Graphical solution of the problem. Graphs of supply and demand curves will be plotted in Microsoft Excel in the domain of economic values: $q \geq 0$, $p \geq 0$. Let's find the maximum possible quantity demanded from the equation $D(q) = 0$. The equation has a root $q = 30$. Thus, the domain of economic values of demand: $0 \leq q \leq 30$.

Let's prepare the data in Microsoft Excel (Fig. 22) to plot the supply and demand curves. We enter the values of the quantity q from 0 to 30 with step 5 in column A. The step is chosen so that the analytically found equilibrium sales volume is a multiple of the step. Enter the formula =-5*A2+150, which is used to calculate demand, in cell B2. Enter the formula =A2^2/4+A2/2+70 used to calculate supply in cell C2. The values in the remaining cells of columns A and B are obtained by autofilling.

	A	B	C
1	Q	D(Q)	S(Q)
2	0	150	70
3	5	125	78,75
4	10	100	100
5	15	75	133,75
6	20	50	180
7	25	25	238,75
8	30	0	310

Fig. 22. Table for plotting the graph

Select the range of cells A1:C8 to plot the graph. We choose the type of the diagram "Point chart". The graph of supply and demand curves, based on the table data, is shown in Fig. 23.

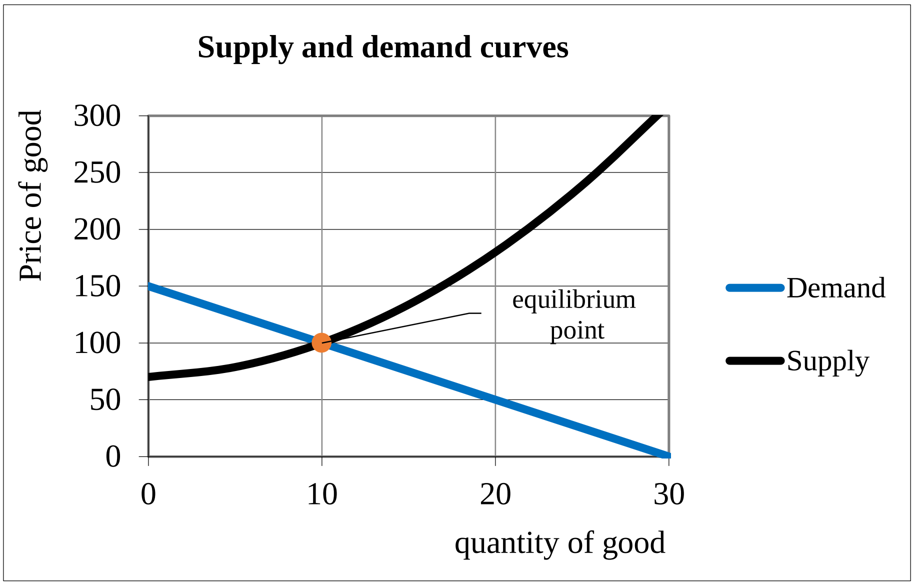

Fig. 23. Equilibrium price and equilibrium sales volume

Determine from the graph the coordinates of the intersection point of the supply and demand curves to find the equilibrium price and equilibrium sales volume: $q = 10$, $p = 100$. So, the equilibrium price is 100 monetary units, the equilibrium sales volume is 10 units.

Consider the impact of price changes on demand, supply, and market equilibrium.

When a seller and a customer enter a market, they study that market and find the demand function for the good and, therefore, the market price. Determining market equilibrium is an important task for the seller and the customer of a good. Suppose that, as a result of studying the market for a certain good and statistical processing of the data obtained, the functions of supply and demand for this good have been given in the following form: demand function is

$$D = 7 - p,$$

supply function is

$$S = p + 1,$$

where D, S are the quantities of good respectively bought and offered for sale per unit of time, p is the price of good.

A linear decreasing function is chosen as the demand function. Demand is described by a decreasing function, because demand decreases as the price rises. The supply function is a linear increasing function: the higher the price in the market for a good, the greater the quantity of the good the firm produces. The use of linear functions of supply and demand is the simplest model of supply and demand. The graphs of the supply and demand curves are shown in Fig. 24.

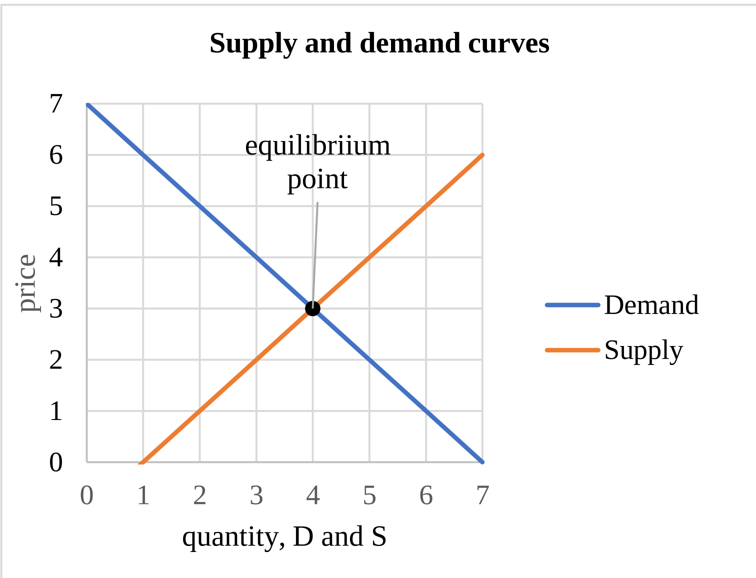

Fig. 24. Supply and demand curves

The optimum for the seller and the customer is equality of supply and demand, and they are equal at the point of intersection of the curves. From the equation $D = S$ we find the equilibrium price. So, the equilibrium price is equal to three monetary units:

$p_0 = 3$. We find the equilibrium sales volume from the supply function $q_0 = S(p_0) = 3 + 1 = 4$. So, the coordinates of the market equilibrium point are $(4;3)$.

The equilibrium price $p_0 = 3$ can be interpreted as a "fair exchange price", which is established as a result of numerous paired transactions between sellers and customers.

Demand equals supply only at this price $p_0 = 3$. The equilibrium is wonderful because demand is fully satisfied, and there is no overproduction of the good and no irrational expenditure of production resources. In this regard, the equilibrium is acceptable and suitable for both groups of participants of market exchange – producers and consumers – and therefore can act as the final goal of the process of regulation by means of prices.

The equilibrium price rationalizes the customer's demand by informing him how much consumption of a given good he can expect; it informs the producer how much good he should produce and deliver to the market.

Any deviation from the equilibrium price activates the forces caused by the laws of the market to return it to its previous equilibrium point. Therefore, the market equilibrium factor is a factor of reducing the indeterminacy of the external environment and limiting risk at the equilibrium point.

The quantity of demand will exceed the quantity of supply $(D > S)$ if the price of goods $p < p_0$. For example, taking $p = 1$, we get

$$D(1) = 7 - 1 = 6,\ S(1) = 1 + 1 = 2,\ \ D(1) - S(1) = 6 - 2 = 4.$$

There is a shortage of goods, equal to the difference between the quantity of good that customers want to buy at a given price (6 units) and the quantity of good that sellers want to sell (2 units). There is excess demand, the market price increases, demand decreases, and the equilibrium point is reached.

If the price of goods $p > p_0$, for example, $p = 5$, the quantity of demand will not reach the quantity of supply $(D < S)$:

$$D(5) = 7 - 5 = 2,\ S(5) = 5 + 1 = 6,\ \ S(5) - D(5) = 6 - 2 = 4.$$

The difference between the quantity of supply and the quantity of demand is excess supply at a given price. Excess supply can cause an increase in inventories and costs. Therefore, until the excess supply is eliminated, the producer (supplier) is forced to reduce prices. This increases the quantity of demand, and the quantity of supply falls, reaching an equilibrium point.

Find how supply and demand will change if the price deviates by a few percent from the equilibrium value. Calculate the price elasticity of demand and the price elasticity of supply:

$$E_p(q) = E_p(7-p) = \frac{-p}{7-p},$$

$$E_p(s) = E_p(p+1) = \frac{p}{p+1}.$$

Let's substitute the value of the equilibrium price $p = 3$ then we get

$$E_{p=3}(q) = -0.75\,, \quad E_{p=2}(s) = 0.75.$$

Since the obtained elasticity coefficients are less than one in absolute value, the demand and supply of this product are price inelastic at the equilibrium price. This means that a change in price will not lead to a sharp change in supply and demand. For example, if the price increases by 1%, demand will decrease by 0.75%, and supply will increase by 0.75%.

Determine the change in revenue when the price increases by 4% of the equilibrium price. We use the formula for the relationship between the demand elasticity and the revenue elasticity:

$$E_p(R) = 1 - \left|E_p(q)\right|.$$

Consequently, the elasticity of revenue at the equilibrium price

$$E_p(R) = 1 - 0.75 = 0.25.$$

Let's find the percentage change in revenue using the price elasticity of revenue

$$E_p(R) \approx \frac{\%\Delta R}{\%\Delta p},$$

then $\%\Delta R \approx E_p(R) \cdot \%\Delta p = 0.25 \cdot 4\% = 1\%$. So, total revenue will increase by 1% if the price increases by 4% of the equilibrium price.

2.4. Cobweb Model of Market Equilibrium

As a rule, the achievement of equilibrium in a competitive economics without collusion (coalitions) is a spontaneous process based on the fact that when the price exceeds the equilibrium price, the quantity of good offered by sellers (producers) will exceed the quantity demanded by customers (consumers), there is downward pressure on the price in this case. The price below the equilibrium level is under upward pressure.

The process of reaching an equilibrium in time is described by a sequence of small discrete steps – "trading days" with numbers $t,\ t+1,\ t+2, \ldots$

Let the supply S_t is given on trading day t, and it determines the price P_t as the solution of the equation $S(P_t) = S_t$.

This price determines the quantity of demand $D_t = D(P_t)$, and the supply for the next trading day is directly oriented to the demand of the previous day

$$S_{t+1} = D_t.$$

Let's show the process of reaching equilibrium using supply and demand curves

$$D = \frac{p+8}{p+2},$$

$$S = p + 0.5.$$

Consider the algorithm for calculations.

Step 1 (Day 1): Let the supply $S_1 = 1.5$. Then we find the price from the equation $S_1 = p_1 + 0.5$. Get $p_1 = S_1 - 0.5 = 1$. This price determines the quantity of demand $D_1 = \frac{p_1+8}{p_1+2} = 3$. Deviation of demand from supply is $E_1 = D_1 - S_1 = 1.5$.

Step 2 (Day 2): The supply for the next trading day is oriented to the demand of the previous day: $S_2 = D_1 = 3$. Find the corresponding price $p_2 = S_2 - 0.5 = 2.5$. This price determines the quantity of demand $D_2 = \frac{p_2+8}{p_2+2} = 2.33$. Deviation of demand from supply is $E_2 = D_2 - S_2 = -0.67$.

The next Step (The next day). Repeat Step 2.

Let's continue the calculations in the Microsoft Excel spreadsheet (Fig. 25). Column A is the number of the day. Columns B, C, D are supply, price, and demand, respectively. Column E is the difference between supply and demand. Let's enter the initial quantity demanded (1.5) in cell B2. We calculate the corresponding price using the formula =B2-0,5 in cell C2. We enter the formula for calculating the quantity demanded in cell D2: =(C2+8)/(C2+2). The formula =D2-B2 is entered in cell E2. The next day's supply in cell B3 is the demand of the previous day: =D2. The values in the other cells are obtained by autofilling.

	A	B	C	D	E
1	*t*	*S*	*P*	*D*	*E=D-S*
2	1	1,5	1	3	1,5
3	2	3	2,5	2,33	-0,67
4	3	2,33	1,83	2,57	0,23
5	4	2,57	2,07	2,48	-0,09
6	5	2,48	1,98	2,51	0,03
7	6	2,51	2,01	2,50	-0,01
8	7	2,50	2,00	2,50	0,00

Fig. 25. Calculation of the convergence to market equilibrium

On the 7th "market day" the price setting process converges to an equilibrium price $P_0 = 2$. The convergence of the price to the equilibrium value, based on these calculations, is shown in Fig. 26.

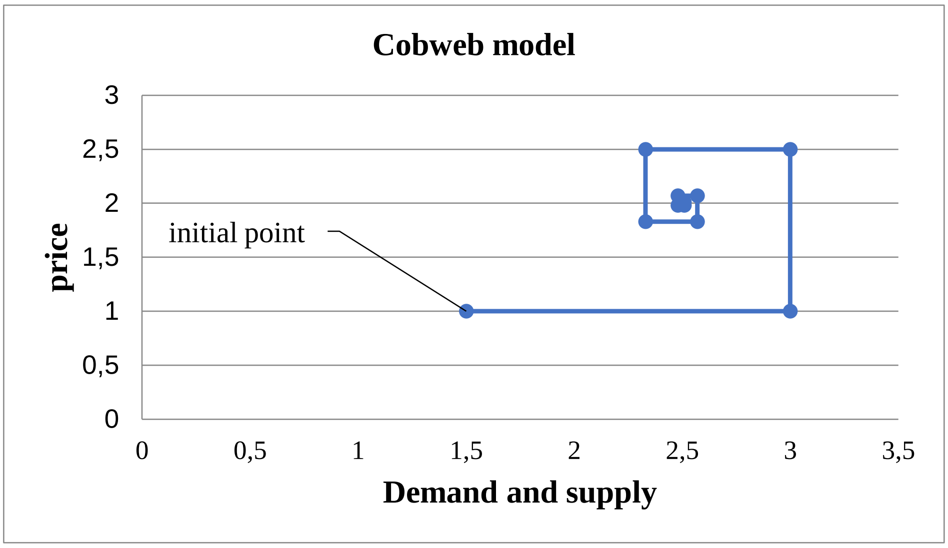

Fig. 26. Cobweb model

The geometric illustration of this process of approaching equilibrium is similar to a cobweb pattern, and therefore the model is often called a cobweb model.

The convergence of the model is guaranteed under the condition

$$S'(P) > |D'(P)|.$$

This condition means that producers react more powerfully to price changes than consumers. Possible variants of convergence (divergence) are shown in fig. 27.

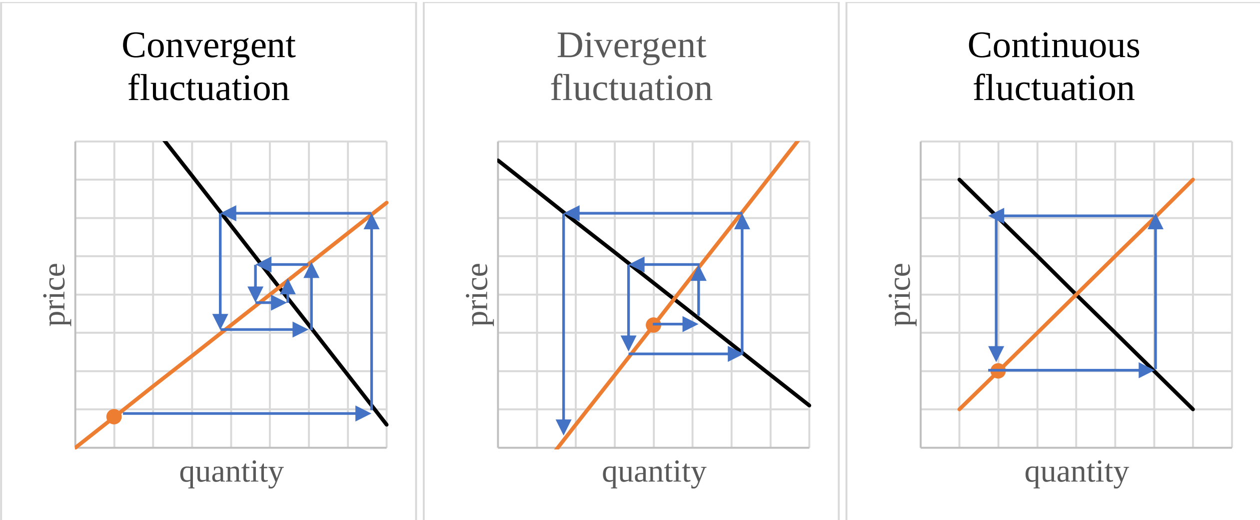

Fig. 27. Variants of convergence (divergence)

3. MODELS OF CONSUMER BEHAVIOR

Consider mathematical models of microeconomics – models of consumer behavior. The main problem in the study of consumer behavior is to find out how many goods and services they buy at a given prices and income.

3.1. Utility Function and Consumer Behavior Model

Let assume that the consumer has income *I*, which he spends entirely on the purchase of goods (services). The prices of the goods are given. Taking into account prices, income and own preferences and tastes, the consumer buys certain quantities of goods, a mathematical model of such consumer behavior is called a consumer choice model.

Consumption bundle *x* is the collection of all the goods that the consumer can presumably buy. The consumption bundle is a vector $x = (x_1, x_2, \dots, x_n)$, the coordinate x_1 is the number of units of the first good; the coordinate x_2 is the number of units of the second good, and the coordinate x_n is the number of units of the *n*-th good:

$$x = (x_1, x_2, \dots, x_n) - \textit{consumption bundle.}$$

The consumer chooses goods based on his or her own preferences and tastes. Consumption theory assumes that the consumer can say of every two bundles that either one is more desirable than the other; or the consumer sees no difference between them.

Utility function is an important concept that measures preferences over a collection of goods and services. Utility represents the satisfaction that consumers receive for choosing and consuming a product or service. Let's denote by $u(x)$ the utility function. The utility function gives each consumption bundle *x* a numerical value that represents the level of satisfaction an individual receives if he buys or consumes this bundle:

$$u: \quad x \quad \to \quad numerical\ value\ of\ satisfaction\ level.$$

Everyone has their own personal tastes and preferences. Therefore, consumer preferences are described by a personal utility function. If bundle A is preferable to bundle *B*, then the utility of bundle *A* is greater: $u(A) > u(B)$. If the consumer receives the same level of satisfaction, then $u(A) = u(B)$.

Indifference curve. Each consumption bundle $x = (x_1, \ldots, x_n)$ corresponds to a point in *n*-dimensional space. If we connect the points corresponding to sets with the same level of utility, we get a curve. This curve is called the indifference curve or the level line of the utility function. The indifference curve is given by the equation

$$u(x_1, \ldots, x_n) = C = const,$$

where *C* is a given level of utility. Any two points on this curve (consumption bundles) are indifferent to the consumer, the consumer receives the same level of satisfaction by purchasing one or the other bundle. An indifference curve shows combinations of goods that provide an equal level of utility or satisfaction.

Let's focus on the simplest example where the consumer chooses between two goods, 1 and 2, and let the quantities chosen for each be x_1 and x_2 respectively. Graphs of indifference curves $u(x_1, x_2) = C_i = const$ corresponding to different values of utility can be plotted in this case in the Cartesian coordinate system $x_1 O x_2$ (fig. 28).

The higher and more rightward the indifference curve lies, the greater the level of utility to which it corresponds. Indifference curves are downward sloping from left to right, they are convex with respect to the origin.

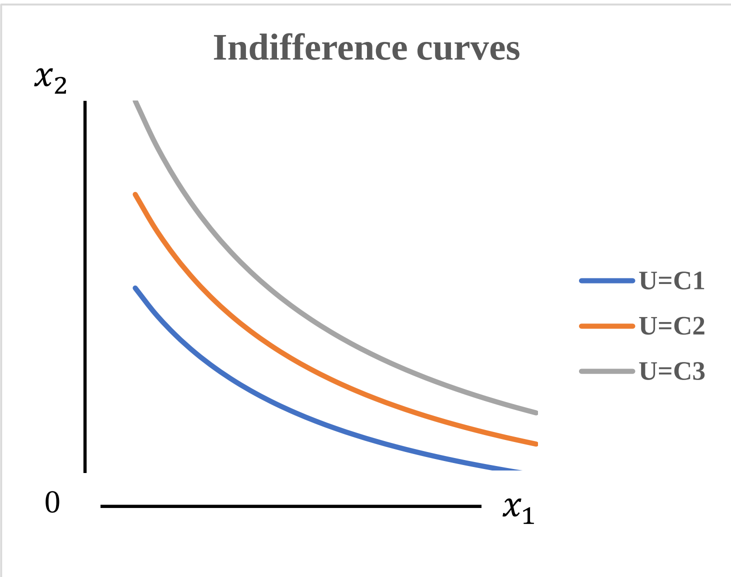

Fig. 28. Indifference Curves corresponding to utility levels $C_1 < C_2 < C_3$

Consumption theory assumes that the consumer is trying to maximize utility or satisfaction within a limited budget. **The consumer choice problem** is to choose such a consumption bundle $(x_1, \ldots, x_n)$, which maximizes the utility given the budget constraint.

A budget constraint means that the cost of consumer's consumption bundle cannot exceed the consumer's income:

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \le I \text{ ,}$$

where $p_1, \ldots, p_n$ are the prices of one unit of the first, second, ..., *n*-th good, respectively, and *I* is the consumer's income, which he is willing to spend to buy a collection of *n* goods. The values of $p_1, \ldots, p_n$ and *I* are given. The consumer can only purchase as much as his income will allow; hence he is limited by his budget.

The consumer choice problem has the form:

$$max\, u\,(x_1, \ldots, x_n)$$

under the conditions

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \le I \text{ ,}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ \ldots,\ x_n \ge 0.$$

The bundle $({x_1}^*, \ldots, {x_n}^*)$, which is the solution of the consumer choice problem, is called the optimal consumption bundle.

3.2. Consumer Choice Problem in the Two-Good Case

Let the consumer make a choice between two goods. The consumer choice problem takes the following form:

$$max\, u\,(x_1, x_2) \tag{3.1}$$

$$p_1 x_1 + p_2 x_2 \leq I\,, \tag{3.2}$$

$$x_1 \geq 0,\;\; x_2 \geq 0. \tag{3.3}$$

Graphical analysis. Let find the budget set. **The budget set** is the set of combinations of goods available for consumption. It is defined by inequalities (3.2) – (3.3). Let us draw a budget line. The equation of the budget line has the form: $p_1 x_1 + p_2 x_2 = I$. It is most convenient to draw the budget straight line through the points of intersection with the coordinate axes, where all income is spent on one product: $(I/p_1\,, 0)$ and $(0, I/p_2)$. The budget straight line is shown in Fig. 29.

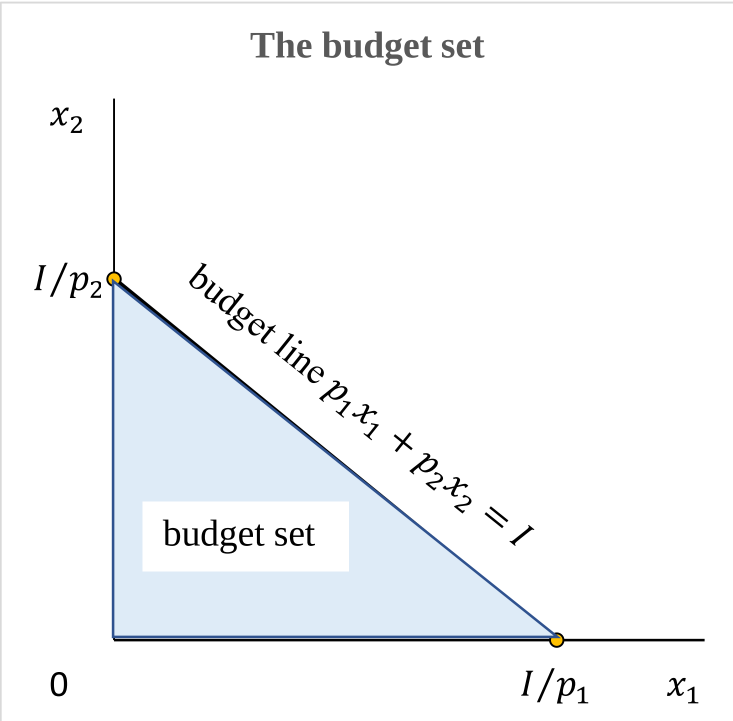

Fig. 29. The budget line and the budget set

The budget set is a triangle bounded by the coordinate axes and the budget line (Fig. 29). Budget set represents all the combinations of goods and services that a consumer may purchase given current prices within his or her given income.

The consumer chooses the most preferred point from the budget set. The question is, how is this point chosen? How do we know how much good 1 (x_1) and how much of good 2 (x_2) to consume?

Among all the points of the budget set it is required to find the point belonging to the indifference curve with the maximum level of utility. Finding this point can be interpreted graphically as a sequential transition to curves of higher and higher levels of utility as long as these curves still have common points with the budget set (fig. 30).

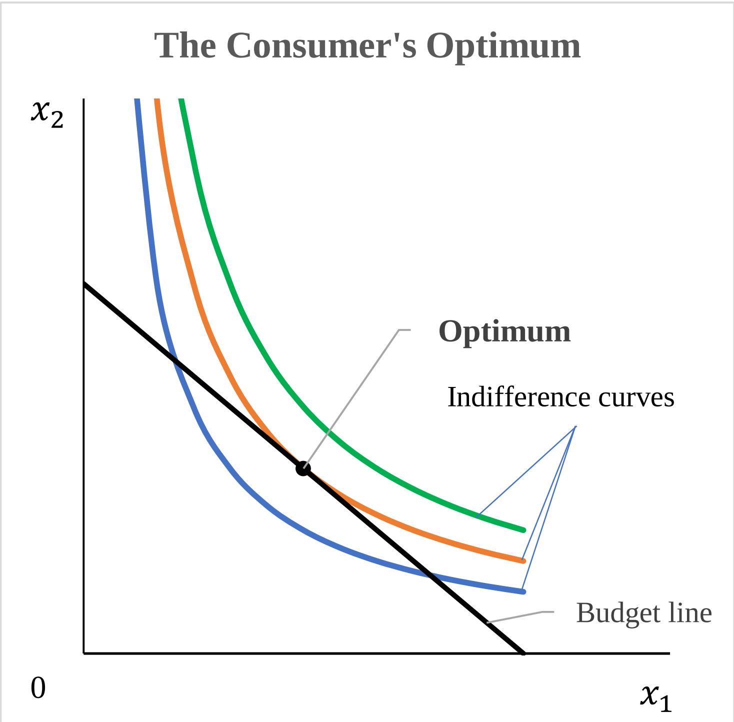

Fig. 30. The Consumer's Optimum

Geometric interpretation of the solution. The solution of the consumer choice problem or the optimal consumption bundle $(x_1{}^*, x_2{}^*)$ is the point where the indifference curve touches the budget line $p_1x_1 + p_2x_2 = I$ (Fig. 30).

Formal analysis. Let's assume that the conditions of non-negativity of variables $x_1 \geq 0,\ x_2 \geq 0$ are automatically fulfilled at the optimal point $(x_1{}^*, x_2{}^*)$, derived from the properties of the utility function $u(x_1, x_2)$.

Consider the problem on conditional extremum:

$$max\, u\,(x_1, x_2)$$

under the condition

$$p_1x_1 + p_2x_2 \leq I.$$

The consumer's objective function $u(x_1, x_2)$ is maximized subject to the budget constraint. Let use the Lagrange multiplier technique to solve this problem. The Lagrange multiplier technique finds the maximum or minimum of a multivariable function when there is some constraint on variables. The Lagrangean function for this optimization is:

$$L(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(p_1x_1 + p_2x_2 - I).$$

Take the first order partial derivatives of the Lagrange function with respect to the variables x_1, x_2 and λ, equate these partial derivatives to zero. We get the system of equations

$$\frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1} + \lambda p_1 = u_1^{'} + \lambda p_1 = 0, \quad (3.4)$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} + \lambda p_2 = u_2^{'} + \lambda p_2 = 0, \quad (3.5)$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 - I = 0. \quad (3.6)$$

where $\frac{\partial u}{\partial x_1} = u_1'$ is the first order partial derivative of the utility function with respect to the variable x_1 and $\frac{\partial u}{\partial x_2} = u_2'$ is the first order partial derivative of the utility function with respect to the variable x_2.

The first partial derivatives of the utility function are called the marginal utility of the products:

$$\frac{\partial u}{\partial x_i} = u_i^{'} \equiv Mu_i > 0.$$

The marginal utility of the *i*-th good shows by how much the utility increases if the consumption of the good increases by a unit.

Getting rid of the variable λ in the system of equations (3.4)–(3.6), we obtain the system of two equations with two unknowns x_1, x_2:

$$\frac{u_1^{'}}{u_2^{'}} = \frac{p_1}{p_2}, \quad (3.7)$$

$$p_1x_1 + p_2x_2 = I. \quad (3.8)$$

Solving the system (3.7)–(3.8), we find the optimal consumption bundle $({x_1}^*, {x_2}^*)$.

Substituting the solution $({x_1}^*, {x_2}^*)$ to the equality (3.7), we obtain that the ratio of marginal utilities of goods is equal to the ratio of prices p_1 and p_2 for these goods at the point of optimum:

$$\frac{u_1'({x_1}^*, {x_2}^*)}{u_2'({x_1}^*, {x_2}^*)} = \frac{p_1}{p_2}.$$

Example 1. The marginal utility of butter for a Frenchman depends on the quantity of butter: $Mu_1 = 40 - 5x_1$. The marginal utility of baguette is: $Mu_2 = 20 - 3x_2$. The price of a kilogram of butter is 5 francs, the price of a baguette is 1 franc. The total income of the consumer is 20 francs per week. How much baguettes and butter should a Frenchman consume to get maximum satisfaction?

Solving: $p_1 = 5$, $p_2 = 1$, $I = 20$. The optimal solution is found from the system of equations (3.7)–(3.8), which for this problem has the form:

$$\begin{cases} \dfrac{40 - 5x_1}{20 - 3x_2} = \dfrac{5}{1} \\ 5x_1 + x_2 = 20 \end{cases}.$$

The system of equations can be written as

$$\begin{cases} 40 - 5x_1 = 5(20 - 3x_2) \\ x_2 = 20 - 5x_1 \end{cases}.$$

Solving a system of linear algebraic equations, we obtain $x_1 = 3$, $x_2 = 5$. So, a Frenchman consumes 3 kilograms of butter and 5 baguettes a week.

Example 2. Solve the consumer choice problem in the case when the prices of goods are $p_1 = 10$, $p_2 = 2$, the income of the consumer is equal to $I = 60$, and the utility function has the form: $u(x_1, x_2) = x_1 x_2$.

Solving: The solution of the system of equations (3.7)–(3.8) in the case of a given utility function is

$$x_1 = \frac{I}{2p_1} = 3, \qquad x_2 = \frac{I}{2p_2} = 15.$$

Graphic solution: The budget straight line has the form: $10x_1 + 2x_2 = 60$. Let us write this equation as follows: $x_2 = 30 - 5x_1$. The indifference curve $u(x_1, x_2) =$

$x_1 x_2 = C$ passing through the optimum point $(3, 15)$ has the equation $x_1 x_2 = 45$ $(x_2 = 45/x_1)$, since $u(3,15) = 45 = C$.

Let's plot the budget straight line and the indifference curve in Microsoft Excel, using the spreadsheet (Fig. 31). Column A is the number of the first good. The formula =30-5*A2 is entered in the cell B2. The cell C3 contains a formula: =45/A3. The values in the other cells of columns B and C are obtained by autofilling.

	A	B	C
1	x_1	budget line	indifference curve
2	0	30	
3	1	25	45
4	2	20	22,5
5	3	15	15
6	4	10	11,25
7	5	5	9
8	6	0	7,5

Fig. 31. Spreadsheet for plotting graphs

The budget line and the indifference line are shown in the fig. 32. We can see that the optimal consumption bundle (3;15) is the point of tangency of the indifference curve with the budget line.

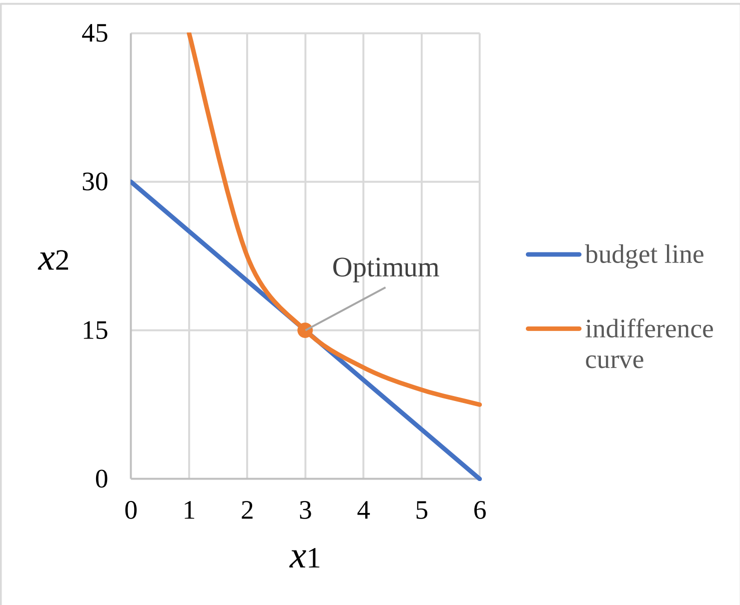

Fig. 32. The Consumer's Optimum

3.3. Responses to Price Changes: Substitution and Income Effects

Determination of substitutability of goods. If an increase in the price of one good leads to a decrease in demand for that good and an increase in demand for another good, then these goods are substitutable. An increase in the price of a good will increase demand for its substitutes, while a decrease in the price of a good will decrease demand for its substitutes. If an increase in the price of one good leads to a decrease in demand for another good, then these goods are complementary.

Real substitutability can be distorted by a general decline in consumer welfare when the price of a good rises: another good can replace a higher-priced good, but demand for it may not rise because the overall welfare of the consumer has declined. The concept of **the income compensated price change** is introduced to remove this distortion. It is such an increase in price that is accompanied by an increase in the consumer's income, allowing him to receive the original level of welfare.

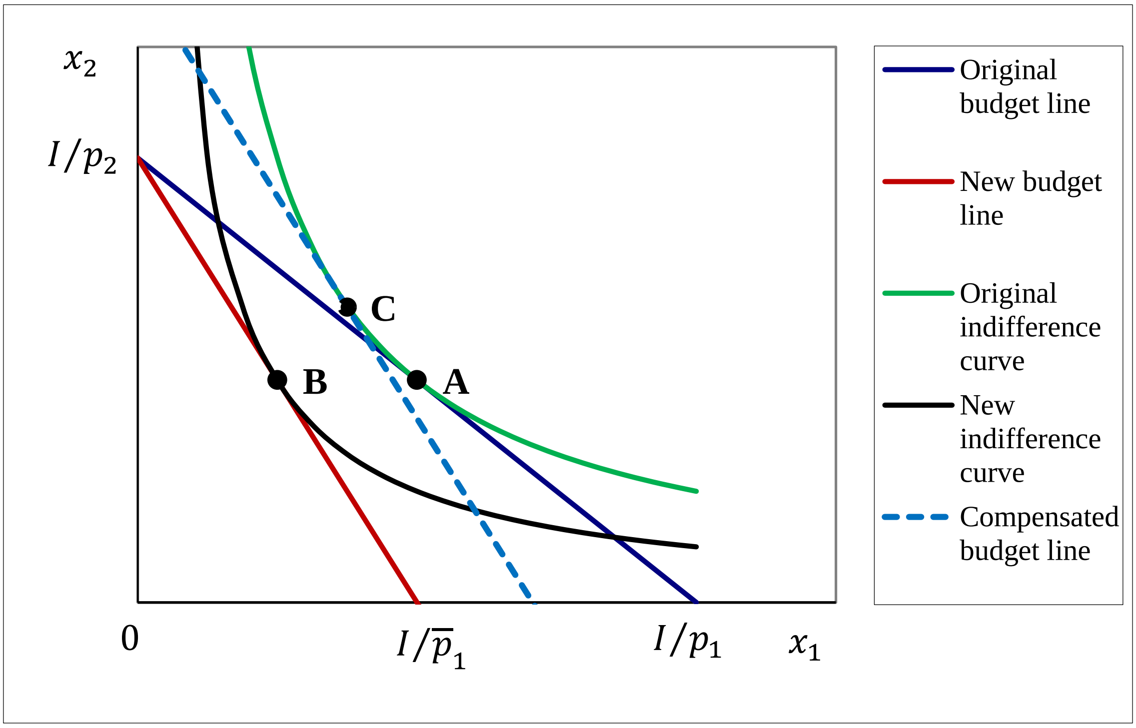

Fig. 33. Responses to price changes: Substitution and Income effects

Consider geometric illustration of the income compensated price change (fig. 33). Let p_1, p_2 are the original market prices of one unit of the first and second good, respectively, and I is the income of the consumer. The original budget line (dark blue line in the fig. 33) passes through the points of intersection with the coordinate axes: $(I/p_1, 0)$ and $(0, I/p_2)$. Point A is the original optimum point, it is the point of tangency between the indifference curve, corresponding to original level of utility C, and the budget line.

Let the price of the first good increases. Let $\overline{p}_1$ is new price of the first good, $\overline{p}_1 > p_1$. Then the new budget line (red line in the fig. 33) will pass through the points $\left(I/\overline{p}_1, 0\right)$ and $(0, I/p_2)$. A higher price for a good will cause the budget line to shift to the left, so that it is tangent to a lower indifference curve representing a reduced level of utility. The optimum point will change. The new optimum point *B* is the point where the indifference curve touches the new budget line. This indifference curve corresponds to a lower utility level.

Let increase the consumer's income so as to compensate for the loss of utility. A rise in income causes the budget line to shift to the right. Let the consumer's income increases so that the budget line (dashed line parallel to red line) touches the original indifference curve at some point *C* (fig. 33).

The movement from the original optimum point (*A*) to point *C* is a **substitution effect**; it shows change in demand when an increase in price is accompanied by the preservation of the original level of welfare. The substitution effect will encourage people to consume less of the good, which has become more expensive, and to consume more of the substitute good.

The income effect is the movement from point *C* to *B*, which shows how consumer reacts to a reduction in his income, but holding constant the relative prices (because the dashed line has the same slope as the new budget line). In this case, where the price of one good increases, buying power is reduced from the higher indifference curve to the lower indifference curve. So, the income effect means that consumption of both goods should fall.

The movement from the original optimum point (*A*) to point *B* is **the total result of price growth** (with no income compensation).

Example. A consumer's preferences are described by a utility function $u(x_1, x_2) = x_1 x_2$, where x_1 is quantity of consumption of bananas, x_2 is quantity of consumption of Pepsi-Cola. The price of 1 kilogram of bananas is 25 rubles, 1 liter of Pepsi-Cola costs 20 rubles. The consumer spent 200 rubles a week on these goods in the summer. The price of bananas rose to 50 rubles per kilogram in winter, the price of Pepsi-Cola did not change. Find out:

1) the optimal consumption of bananas and Pepsi-Cola in summer;

2) the optimal consumption of bananas and Pepsi-Cola in winter;

3) the amount of costs needed in winter to achieve the same level of utility as in summer.

Let's introduce the notation: $u(x_1, x_2) = x_1 x_2$, $p_1 = 25$, $p_2 = 20$, $I = 200$, $\bar{p}_1 = 50$.

Problem solving:

1) Let's find the optimal consumption in summer.

We use the system of equations (3.7)–(3.8) for finding the optimal consumption bundle. Take the partial derivatives of the utility function and substitute them into the equation (3.7). We obtain the system of equations:

$$\frac{x_2}{x_1} = \frac{p_1}{p_2},$$

$$p_1 x_1 + p_2 x_2 = I.$$

The first condition means that the amount of money spent on each good must be the same: $p_1 x_1 = p_2 x_2$. Then we get from the second condition:

$$2p_1 x_1 = I,$$

$$2p_2 x_2 = I.$$

The optimal consumption bundle is as follows:

$$x_1 = \frac{I}{2p_1}, \qquad x_2 = \frac{I}{2p_2}. \tag{3.9}$$

Thus, the cost of buying each good is half of the consumer's total income, and to find the optimal quantity of each good, we must divide the cost of buying it by the price of the good.

The optimal summer consumption:

$$x_1 = \frac{200}{2 \cdot 25} = 4, \quad x_2 = \frac{200}{2 \cdot 20} = 5.$$

The consumer buys 4 kilograms of bananas and 5 liters of Pepsi-Cola a week in summer and receives a level of satisfaction equal to

$$u(x_1, x_2) = x_1 x_2 = 4 \cdot 5 = 20 \text{ units.}$$

2) Let's find the optimal winter consumption using the solution of the consumer choice problem (3.9) and substituting the winter prices of goods in it:

$$x_1 = \frac{I}{2\bar{p}_1}, \quad x_2 = \frac{I}{2p_2}.$$

The optimal winter consumption:

$$x_1 = \frac{200}{2 \cdot 50} = 2, \quad x_2 = \frac{200}{2 \cdot 20} = 5.$$

The consumer buys 2 kilograms of bananas and 5 liters of Pepsi-Cola a week in winter and receives a level of satisfaction equal to

$$u(x_1, x_2) = x_1 x_2 = 2 \cdot 5 = 10 \text{ units.}$$

The quantity of bananas consumed and the level of consumer satisfaction decreased by 2 times compared to the summer.

3) Let's increase the winter cost of consuming bananas and Pepsi-Cola so that the level of utility does not decrease. Let's solve the problem in general form. Suppose the price p_1 has increased by a factor of k, and the consumer receives the necessary income compensation. Let's denote the new income value by $\bar{I}$, the new demands by $\bar{x}_1$ and $\bar{x}_2$. Using the equality of utility levels in summer and winter $u(\bar{x}_1, \bar{x}_2) = u(x_1, x_2)$, taking into account the utility function we get $\bar{x}_1 \bar{x}_2 = x_1 x_2$. Substituting the demand functions (3.9), corresponding to the new price of the first good and the new income, we get the equation

$$\frac{\bar{I}}{2kp_1} \cdot \frac{\bar{I}}{2p_2} = \frac{I}{2p_1} \cdot \frac{I}{2p_2}.$$

From there we find **the value of income with compensation**

$$\bar{I} = \sqrt{k} I \tag{3.10}$$

Corresponding new demand

$$\bar{x}_1 = \frac{\bar{I}}{2kp_1} = \frac{\sqrt{k}I}{2kp_1} = \frac{I}{2\sqrt{k}p_1} = \frac{x_1}{\sqrt{k}}, \quad \bar{x}_2 = \frac{\sqrt{k}I}{2p_2} = \sqrt{k}x_2 .$$

So, the demand for the first good decreases by $\sqrt{k}$ times, and the demand for the second good increases by $\sqrt{k}$ times in the case of increase in price that is accompanied by an increase in the consumer's income:

$$x_1 = \frac{x_1}{\sqrt{k}}, \quad \bar{x}_2 = \sqrt{k}x_2 . \quad (3.11)$$

In the given problem

$$\bar{I} = I\sqrt{2} = 200\sqrt{2} \approx 282.84 \text{ rubles.}$$

The consumer needs 282.84 rubles in winter to receive the same level of utility as in summer. Consumption in this case will be:

$$x_1 = \frac{\bar{I}}{2\bar{p}_1} = \frac{282.84}{2 \cdot 50} \approx 2.83\,, \quad \bar{x}_2 = \frac{\bar{I}}{2p_2} = \frac{282.84}{2 \cdot 20} \approx 7.07.$$

The consumer buys 2.83 kilograms of bananas and 7.07 liters of Pepsi-Cola a week. The consumer will receive a level of satisfaction

$$u(\bar{x}_1, \bar{x}_2) = \bar{x}_1\bar{x}_2 = 2.83 \cdot 7.07 = 20.0081 \approx 20 \text{ units.}$$

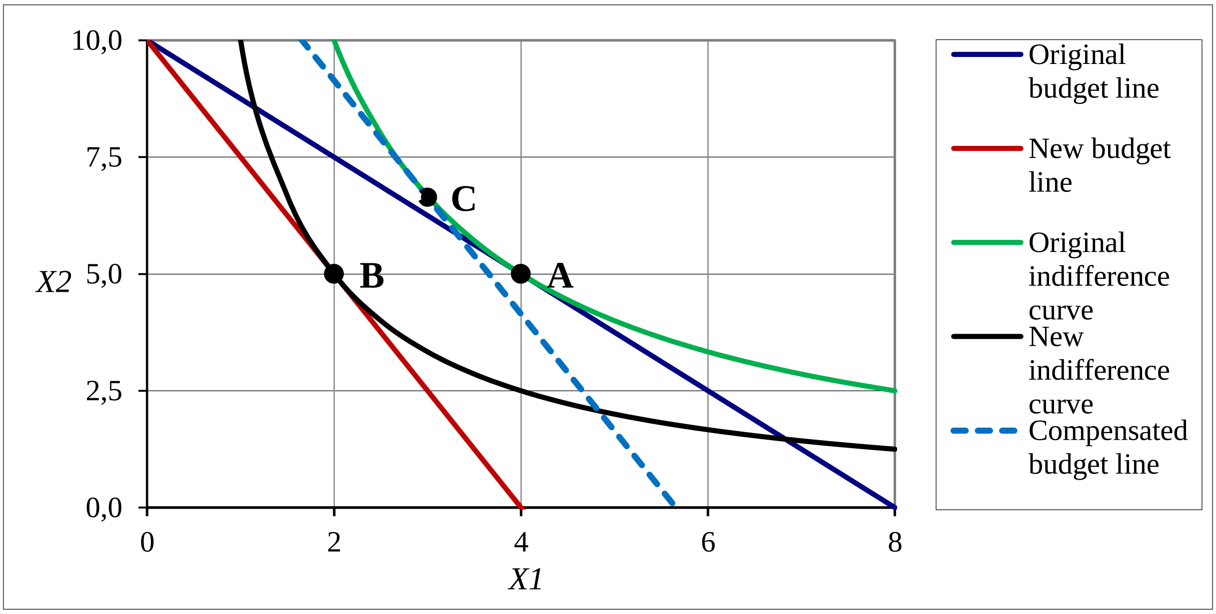

Fig. 34. Substitution and Income effects

Let graphically illustrate the substitution and income effects. The graphs of the original budget line, the budget line after the price increase for the first product (without income compensation and with compensation) and the indifference curves passing through the points of optimum are shown in fig. 34.

The equation of the original budget straight line has the form: $25x_1 + 20x_2 = 200$. The equation of the new budget straight line after the price increase of the first good looks as follows: $50x_1 + 20x_2 = 200$.

The equation of the budgetary straight line as a result of an increase in the price of the first good, which is accompanied by an income increase: $50x_1 + 20x_2 = 282.84$.

The indifference curve passing through the original optimum point (4,5) is given by the equation $x_1 x_2 = 20$. The equation of the indifference curve passing through the new optimum point (2,5) is as follows: $x_1 x_2 = 10$.

Let's prepare a worksheet in Microsoft Excel to plot the graphs given by these equations:

	A	B	C	D	E	F
1	X1	X2=10-1,25X1	X2=10-2,5X1	X2=20/X1	X2=10/X1	X2=14,142-2,5X1
2	0,5	9,4	8,8	40,0	20,0	12,9
3	1,0	8,8	7,5	20,0	10,0	11,6
4	1,5	8,1	6,3	13,3	6,7	10,4
5	2,0	7,5	5,0	10,0	5,0	9,1
6	2,5	6,9	3,8	8,0	4,0	7,9
7	3,0	6,3	2,5	6,7	3,3	6,6
8	3,5	5,6	1,3	5,7	2,9	5,4
9	4,0	5,0	0,0	5,0	2,5	4,1
10	4,5	4,4	-1,3	4,4	2,2	2,9
11	5,0	3,8	-2,5	4,0	2,0	1,6
12	5,5	3,1	-3,8	3,6	1,8	0,4
13	6,0	2,5	-5,0	3,3	1,7	-0,9
14	6,5	1,9	-6,3	3,1	1,5	-2,1
15	7,0	1,3	-7,5	2,9	1,4	-3,4
16	7,5	0,6	-8,8	2,7	1,3	-4,6
17	8,0	0,0	-10,0	2,5	1,3	-5,9

Fig. 35. Worksheet for graph plotting

The original choice is *A*, the point of tangency between the original budget constraint and indifference curve. The new choice is *B*, the point of tangency between the new budget constraint and the lower indifference curve. Point *C* is the point of tangency between the dashed line, where the slope shows the new higher price of bananas, and the original indifference curve. The substitution effect is the shift from *A* to *C*, which means getting fewer bananas and more Pepsi-Cola. The income effect is the shift from *C* to *B*; it means the reduction in buying power that causes a shift from the higher indifference curve to the lower indifference curve, with relative prices remaining unchanged. The income effect results in less consumed of both goods. Both substitution and income effects cause fewer bananas to be consumed.

3.4. Stone's Consumer Choice Model

Consider the consumer choice problem in the case of a utility function called the Richard Stone function. This function has the form:

$$u(x) = (x_1 - a_1)^{\alpha_1}(x_2 - a_2)^{\alpha_2}\dots(x_n - a_n)^{\alpha_n}\,. \tag{3.12}$$

Here a_i is the minimum quantity of the good i, which is purchased in any case and is not a matter of choice. The collection of goods $(a_1,\dots,a_n)$ is the minimum consumer basket. Parameters $\alpha_i > 0$ show the relative values of goods to the consumer. Relative value is attractiveness measured in terms of utility of one good relative to another.

Adding to the objective function (3.12) the budget constraint and non-negativity conditions, we obtain a problem called Richard Stone's model

$$max\, u\,(x) = (x_1 - a_1)^{\alpha_1}(x_2 - a_2)^{\alpha_2}\dots(x_n - a_n)^{\alpha_n},$$

$$p_1x_1 + p_2x_2 + \dots + p_nx_n \le I,$$

$$x_1 \ge 0,\, x_2 \ge 0, \dots, x_n \ge 0.$$

To solve the problem let's form a Lagrangian

$$L(x) = u(x) + \lambda(p_1x_1 + p_2x_2 + \dots + p_nx_n - I).$$

Take the first order partial derivatives of the utility function with respect to variables $x_1,\dots,x_n$ and λ, and equate them to zero. We obtain the system of equations

$$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} + \lambda p_i = 0, \qquad i = 1,2,\dots,n, \tag{3.13}$$

$$\frac{\partial L}{\partial \lambda} = p_1x_1 + \dots + p_nx_n - I = 0. \tag{3.14}$$

Take the derivative

$$\frac{\partial u}{\partial x_i} = \alpha_i (x_1 - a_1)^{\alpha_1} \ldots (x_i - a_i)^{\alpha_i - 1} \ldots (x_n - a_n)^{\alpha_n} = \frac{\alpha_i u(x)}{x_i - a_i}.$$

Then the system of equations (3.13) – (3.14) takes the form

$$\frac{\alpha_i u(x)}{x_i - a_i} + \lambda p_i = 0, \quad i = 1,2,\ldots,n, \tag{3.15}$$

$$p_1 x_1 + \ldots + p_n x_n = I. \tag{3.16}$$

We isolate x_i then the equation (3.15) has the form

$$x_i = a_i - \frac{\alpha_i u(x)}{\lambda p_i} \tag{3.17}$$

Multiply each i-th equation (3.17) by p_i and sum them, we obtain

$$\sum_i p_i x_i = \sum_i p_i a_i - \frac{u(x) \sum \alpha_i}{\lambda}.$$

Since the budget constraint is satisfied as an equality at the optimum point, substitute $\sum_i p_i x_i$ for I. Then we get

$$-\frac{u(x)}{\lambda} = \frac{I - \sum p_i \alpha_i}{\sum \alpha_i}.$$

Substituting this expression into (3.17), we obtain the demand function:

$$x_i = a_i + \frac{\alpha_i (I - \sum p_i \alpha_i)}{p_i \sum \alpha_i} \tag{3.18}$$

Formula (3.18) is the solution of Richard Stone's model. Consider the interpretation of this formula. At first, the minimum necessary number of each good a_i is purchased. Then the amount of money remaining after that $I - \sum p_i \alpha_i$ is calculated, which is divided in proportion to the relative values of goods α_i. Dividing the amount of money $\alpha_i (I - \sum p_i \alpha_i)/\sum \alpha_i$ assigned to the purchase of the good *i* by the price p_i, we get an additional quantity of the good *i* purchased in excess of the minimum, and add it to a_i.

Example 1. Ann's utility function is $u(x_1, x_2) = {x_1}^{1/3} {x_2}^{2/3}$, where x_1 is number of oranges, x_2 is number of apples. She has an income of 40 monetary units and can buy oranges at 10 monetary units per kilogram and apples at 2 monetary units per kilogram. Find for her optimal values x_1 and x_2 as function of the prices p_1 and p_2 of these goods and income I. What is her optimal bundle?

Solving: Let's use R. Stone's model, because Anna's utility function is a special case of the Stone's function when $a_1 = 0,\ a_2 = 0,\ \alpha_1 = 1/3$, $\alpha_2 = 2/3$. Substituting the values of these parameters into the formula (3.18), we obtain the optimal values x_1 and x_2 as function of the prices p_1 and p_2 of these goods and income I (they are demand functions):

$$x_1 = \frac{\frac{1}{3}I}{p_1\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{I}{3p_1},$$

$$x_2 = \frac{\frac{2}{3}I}{p_2\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{2I}{3p_2}.$$

These demand functions mean that Anna assigns a third of her income to buy apples and two thirds of her income to buy oranges. To find what number of oranges and number of apples Anna buys to maximize her utility, let's substitute the prices $p_1 = 10$, $p_2 = 2$ and Anna's income $I = 60$ into the demand functions, we obtain

$$x_1 = \frac{I}{3p_1} = \frac{60}{30} = 2,\quad x_2 = \frac{2I}{3p_2} = \frac{120}{6} = 20.$$

So, Anna buys 2 kilograms of oranges and 20 kilograms of apples.

Example 2. Andy consumers only two goods. The good 1 costs 10 monetary units and the good 2 costs 2 monetary units. His income is 60 monetary units. Andy has a utility function $u(x_1, x_2) = (x_1 - 1)^{1/4}(x_2 - 5)^{3/4}$, where x_1 is number of the good 1, x_2 is number of the good 2. How many goods 1 and how many goods 2 does Andy buy to maximize his utility?

Solving: Andy's utility function is a special case of the Stone's function when $a_1 = 1, a_2 = 5,\ \alpha_1 = 1/4,\ \alpha_2 = 3/4$. Let's substitute these values into the formula (3.18), we obtain the optimal values x_1 and x_2 as demand functions:

$$x_1 = 1 + \frac{\frac{1}{4}(I - p_1 - 5p_2)}{p_1} = 1 + \frac{I - p_1 - 5p_2}{4p_1},$$

$$x_2 = 5 + \frac{\frac{3}{4}(I - p_1 - 5p_2)}{p_2} = 5 + \frac{3(I - p_1 - 5p_2)}{4p_2}.$$

Thus, a quarter of the amount of money remaining after the purchase of the minimum consumer basket is assigned for the purchase of an additional quantity of the good 1. And three quarters of the remaining amount of money is assigned to the purchase of an additional quantity of the good 2. Andy buys 2 units of the good 1 and 20 units of the good 2:

$$x_1 = 1 + \frac{I - p_1 - 5p_2}{4p_1} = 1 + \frac{60 - 10 - 10}{40} = 2,$$

$$x_2 = 5 + \frac{3(I - p_1 - 5p_2)}{4p_2} = 5 + \frac{3(60 - 10 - 10)}{8} = 20.$$

This is Andy's optimal consumption bundle.

4. PRODUCTION MODELS

4.1. Production Functions

A firm uses various resources to produce a product. Resources used in production are called factors of production. These are classified as labor, land and capital. The relationship between the factors of production and output of a firm is called a **production function**.

The production process is seen as a "black box" (fig. 36), the inputs are resources $x_1, \ldots, x_n$, and the output is the result of production (the total product y).

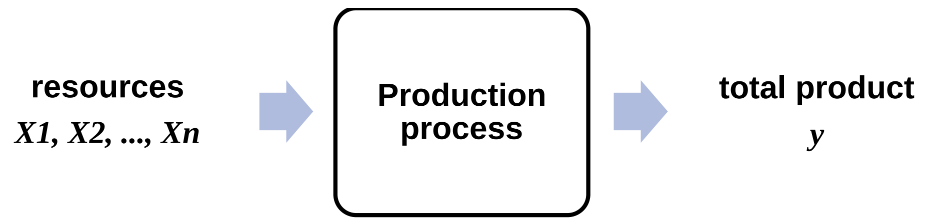

Fig. 36. The concept of the production function

The relationship between inputs and output can be given by function

$$y = f(x_1, \ldots, x_n),$$

it means that output is determined by the quantities of resources used (capital, labor).

The production function represents the technology of a firm, of an industry, or of the economy as a whole. Inputs can be labor time, raw materials, energy, and capital when the production function describes the production of a firm. If the production function describes a region's economy, then capital, labor and natural resources are usually inputs, and the value of the function is the total product (total income) of the region.

Power function is most often used as production function

$$y = a_0 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}.$$

If the exponent degrees of this function are positive and less than 1: $0 < \alpha_i < 1$, then the function corresponds to the economic laws of real productions.

Consider a production function of one variable

$$y = f(x),$$

here x is the amount of one factor of production.

The power function $y = ax^b$ is a typical representative of the class of production functions in the case of a single factor of production. Here a and b are parameters of the production function, these are positive numbers, $b \leq 1$, x is the amount of a factor (for example, amount of labor), y is the product quantity.

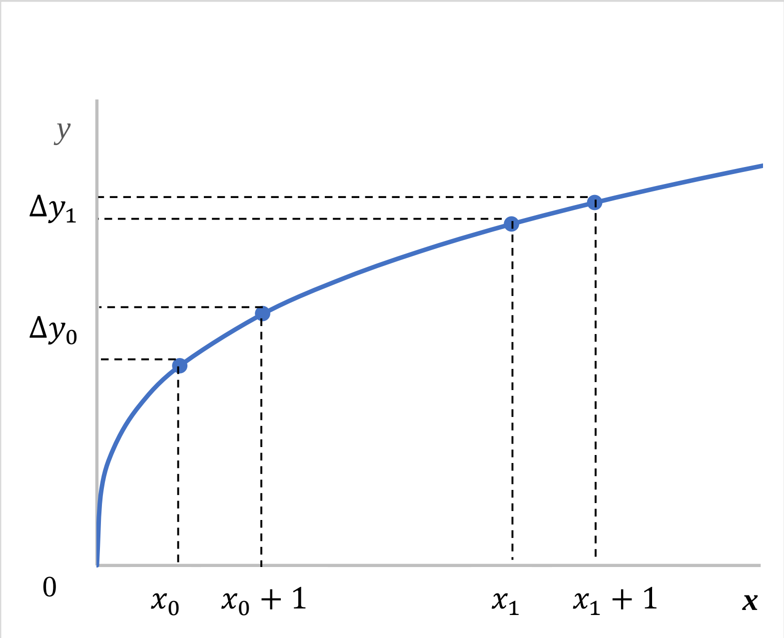

Fig. 37. The graph of the production function $y = ax^b$

The graph of the production function $y = ax^b$ is shown in fig. 37. This is the total product curve that shows the quantities of output that can be obtained from different amounts of a variable factor of production, assuming other factors of production are fixed. The graph shows that quantity of output y increases as the amounts of a factor x increases, but each additional unit of factor gives a smaller increase in quantity of output y. The growth of output y and the decrease in the increment of output y with an increase in the quantity of resource x reflect a fundamental position of economic theory. This is the law of diminishing efficiency.

The slope of a total product curve for any variable factor is a measure of the change in output associated with a change in the amount of the variable factor, with the quantities of all other factors held constant. The amount by which output rises with an additional unit of a variable factor is **the marginal product** of the variable factor. Mathematically, marginal product is the limit of the ratio of the change in output to the change in the amount of a variable factor:

$$MP = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \approx \frac{\Delta y}{\Delta x}.$$

Thus, the marginal product is approximately equal to the slope of the total product curve (Fig. 38).

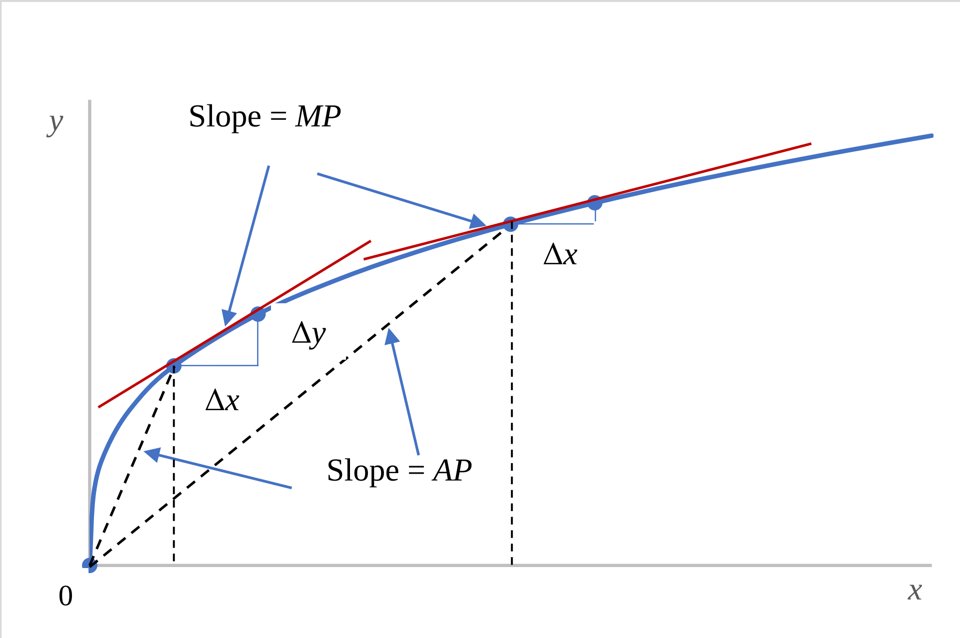

Fig. 38. Marginal and average products

The marginal product of labor MP_L, for example, is the amount by which output rises with an additional unit of labor. It is approximately equal to the ratio of the change in output to the change in the quantity of labor ($\Delta y/\Delta L$), all other factors unchanged. It is measured as the slope of the total product curve for labor:

$$MP_L = \lim_{\Delta L \to 0} \frac{\Delta y}{\Delta L} \approx \frac{\Delta y}{\Delta L}.$$

In addition, we can define **the average product** of a variable factor. It is the output per unit of variable factor:

$$AP = \frac{y}{x}$$

The average product of labor, for example, is the ratio of output to the number of units of labor:

$$AP_L = \frac{y}{L}$$

The average product of labor is the slope of the radius vector drawn from the origin to a point on the total output curve (fig. 38). The concept of average product is often used for comparing productivity levels over time or in comparing productivity levels among nations.

Similarly, we can consider the dependence of the total product on another factor, such as capital. Then **the marginal product of capital and the average product of capital** are determined by the formulas:

$$MP_K = \lim_{\Delta K \to 0} \frac{\Delta y}{\Delta K} \approx \frac{\Delta y}{\Delta K},$$

$$AP_K = \frac{y}{K}.$$

4.2. Substitutability of the Factors of Production

Consider a production function with two variables that describes the dependence of output Y on the invested capital K and labor input L:

$$Y = f(K, L).$$

The following production functions of two variables are used in mathematical modeling:

1. $Y = a_0 K^{a_1} L^{a_2}$ – **Cobb–Douglas production function**.

Here a_0, a_1, a_2 are function parameters. These are positive constants $a_1 > 0$, $a_2 > 0$, also a_1 and a_2 are often such that $a_1 + a_2 = 1$.

2. $Y = AK + BL$ – **Linear production function** (*A*, *B* are constants).

3. $Y = min(aK, bL)$ – **Leontief production function** (*a*, *b* are constants).

The concept called isoquants are used for the analysis of production function with two variable factors. Isoquants are similar to indifference curves of the theory of demand. An isoquant represents all factor combinations which are capable of producing the same level of output.

The isoquants are contour lines which trace the loci of equal outputs. Since an isoquant represents all those combinations of inputs which will be capable of producing an equal quantity of output, the producer would be indifferent between them. Therefore, isoquants are also often called equal product curves or production-indifference curves.

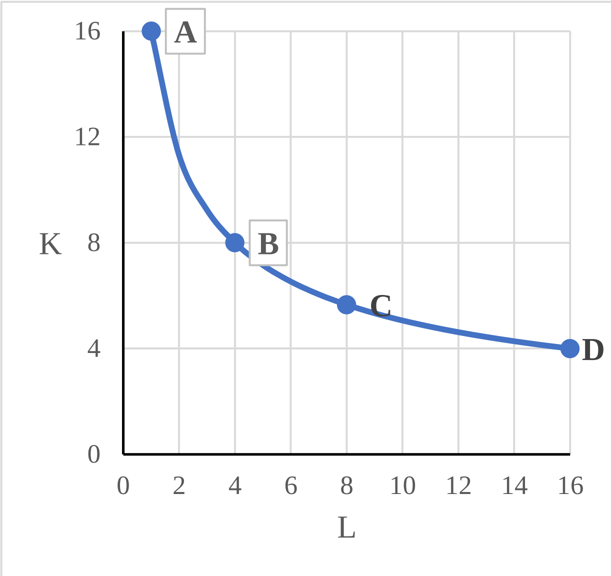

Fig. 39. Isoquant

An isoquant includes all the combinations of factors of production for producing a given level of output. The isoquant of the function $Y = K^{1/2}L^{1/4}$ corresponding to a output equal to 4 units is shown in fig. 39. Each of the factor combinations A, B, C, and D yields the same level of output. Moving down from combination B to combination D, 4 units of capital are substituted by 12 unit of labor in the production process without any change in the level of output.

The isoquant equation has the form

$$Y(K, L) = const.$$

In the case of the Cobb–Douglas production function, the isoquant equation is as follows:

$$a_0K^{a_1}L^{a_2} = Y_0 = const \quad \text{or} \quad K^{a_1} = \frac{Y_0}{a_0}L^{-a_2}.$$

Therefore, the isoquant is a power hyperbola, the asymptotes of which are the coordinate axes. The isoquants of the production functions: the linear function, the Cobb–Douglas function and the Leontief function are shown in fig. 40.

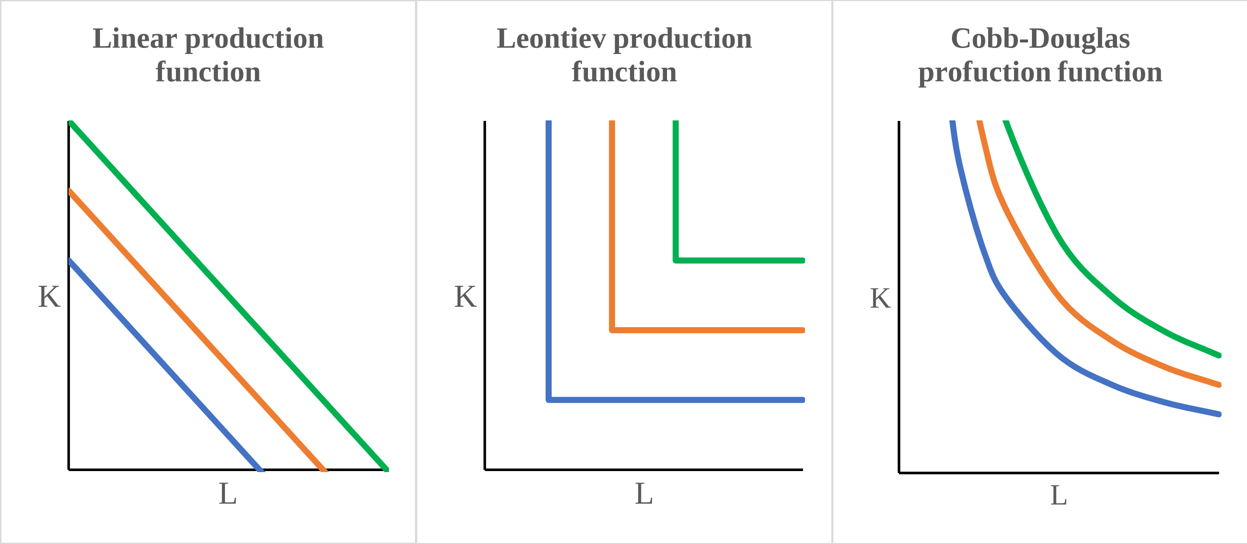

Fig. 40. The isoquants of the production functions

The slope of the isoquant $\partial K/\partial L$ defines the degree of substitutability of the factors of production (fig. 41). The slope of the isoquant decreases (in absolute terms) as we move downwards along the isoquant, showing the increasing difficulty in substituting *K* for *L*. The slope of the isoquant is called the rate of technical substitution, or the **marginal rate of substitution** (*MRS*) of the factors

$$MRS = -\frac{dK}{dL}.$$

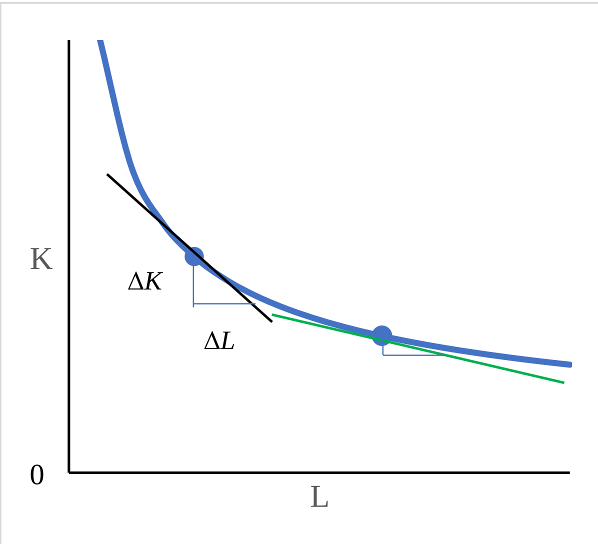

Fig. 41. The slope of the isoquant and substitutability of the factors

Marginal rate of technical substitution of labor for capital may be defined as the number of units of capital which can be replaced by one unit of labor, the level of output remaining unchanged.

It can be proved that the ***MRS* is equal to ratio of the marginal product of the factors**. Since on the isoquant $Y(K, L) = Y_0 = const$, then

$$dY = \frac{\partial Y}{\partial K} dK + \frac{\partial Y}{\partial L} dL = 0.$$

From where

$$MRS = -\frac{dK}{dL} = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{MP_L}{MP_K}.$$

An isoquant includes (is the locus of) all the technically efficient methods (or all the combinations of factors of production) for producing a given level of output. The production isoquant may assume various shapes depending on the degree of substitutability of factors.

Linear isoquant. This is isoquant of the linear production function $Y = AK + BL$. This type assumes **perfect substitutability of factors of production**: a given commodity may be produced by using only capital, or only labor, or by an infinite combination of *K* and *L* (fig. 40).

Input-output isoquant. This is isoquant of the Leontief production function $Y = min(aK, bL)$. This assumes strict complementarity (that is, **zero substitutability**) of the factors of production. There is only one method of production for any one commodity. The isoquant takes the shape of a right angle (fig. 40). This type of isoquant is also called "Leontief isoquant" after Leontief, who invented the input- output analysis.

Smooth, convex isoquant: This is isoquant of the Cobb–Douglas production function $Y = a_0 K^{a_1} L^{a_2}$. This form assumes continuous substitutability of *K* and *L* only over a certain range, beyond which factors cannot substitute each other. The isoquant appears as a smooth curve convex to the origin (fig. 40).

The production function describes not only a single isoquant, but the whole array of isoquants, each of which shows a different level of output. It shows how output varies as the factor inputs change. Production functions involve (and can provide measurements of) concepts which are useful tools in all fields of economics.

Example 1. Consider the production process at a car wash. A worker can wash 16 cars in an 8-hour shift, and an automatic car wash system services 32 cars in 8 hours.

Suppose the car wash should serve 96 cars per shift. This output can be provided by different technological methods (tabl. 7).

Tabl. 7. Technological methods

№	Calculation of production technology	Factors of production
1	$96 : 16 = 6$	only one labor factor is used: 6 workers
2	$96 : 32 = 3$	only one capital factor is used: 3 automatic systems
3	$16 \cdot 4 + 32 \cdot 1 = 96$	4 workers and 1 automatic system are used
4	$16 \cdot 2 + 32 \cdot 2 = 96$	2 workers and 2 automatic system are used

Production in this case is described by a linear production function $Y = 32K + 16L$. The isoquant, which provides output equal to 96 cars, are shown in fig. 42.

This is an example of **perfect substitutability**: it is possible to completely replace workers or automatic car wash systems.

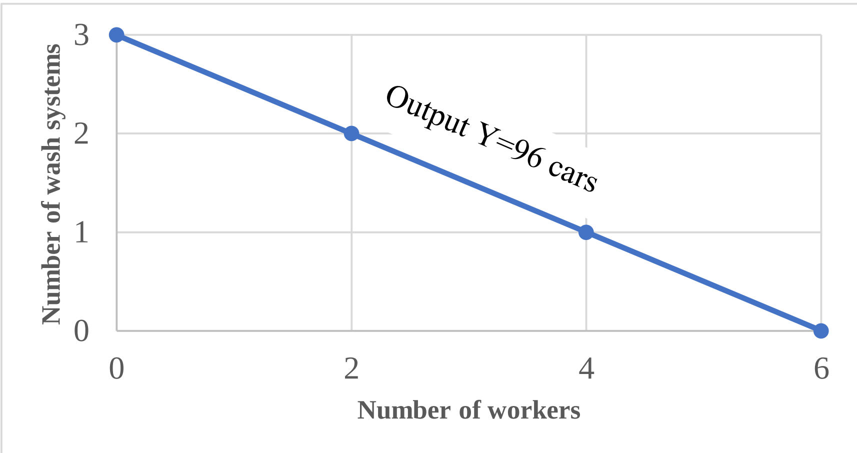

Fig. 42. The isoquant, corresponding to the output of Y=96 cars

Example 2. Consider a car wash that works 16 hours a day. It has 3 washing bays and 4 workers (8-hour shift). Let the washing of the car requires 30 minutes of working time and 30 minutes of occupancy of the washing bay.

The number of cars that can be washed per day depends on which factor is limiting. Let K be the number of wash bays, L be the number of workers.

The bays allow to wash 96 cars: $a \cdot K = \frac{16 \text{ hours}}{0.5} \cdot 3 = 32 \cdot 3 = 96.$

Workers can wash: $b \cdot L = \frac{8 \text{ hours}}{0.5} \cdot 4 = 16 \cdot 4 = 64$ cars.

The production process is described by the Leontief production function

$$Y = min(a \cdot K, b \cdot L) = min(32 \cdot K, 16 \cdot L) = min(96{,}64) = 64.$$

The car wash can service 64 cars per shift, the bays will be idle due to a lack of workers. The limiting factor is labor.

Two washing bays are sufficient for 4 workers:

$$64 \,:\, a = 64 : (16 : 0.5) = 2.$$

The isoquant, which provides output equal to 64 cars, are shown in fig. 43. If there are more than 4 workers in a car wash with 2 bays, the workers will be idle. If there are more than 2 bays at a car wash with 4 workers, the bays will be idle. The output will remain equal to 64 cars per day.

The factors are used in the given proportion in this example: 4 workers for 2 washing bays. Substitution of factors is not possible. A given level of output requires a certain combination of factors. This is an example of zero substitutability or **perfect complementarity** of factors.

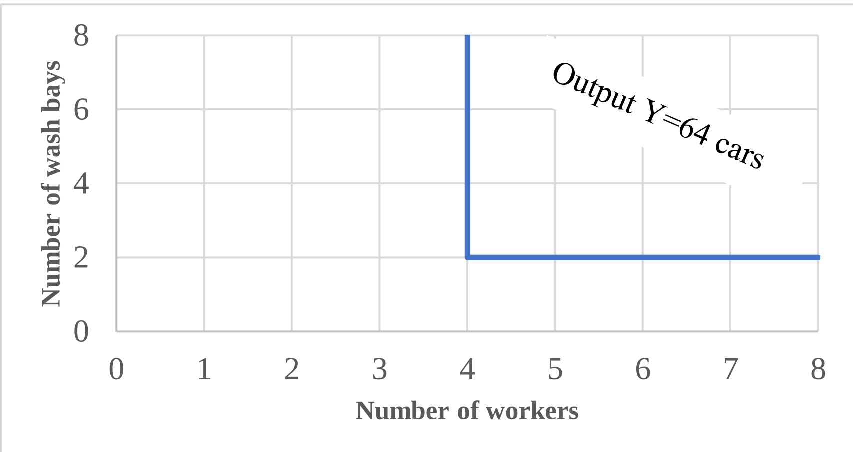

Fig. 43. The isoquant, corresponding to the output of Y=64 cars

The marginal rate of substitution as a measure of the degree of substitutability of factors has a serious defect: it depends on the units of measurement of the factors.

A better measure of the factor substitution is provided by the elasticity of substitution. **The elasticity of substitution** is defined as the limit of the percentage change in the capital labor ratio, divided by the percentage change in the rate of technical substitution (fig. 44)

$$\sigma = E_{MRS}(K/L) = \lim_{\Delta MRS \to 0} \frac{\%\Delta(K/L)}{\%\Delta MRS} \approx \frac{\%\Delta(K/L)}{\%\Delta MRS}.$$

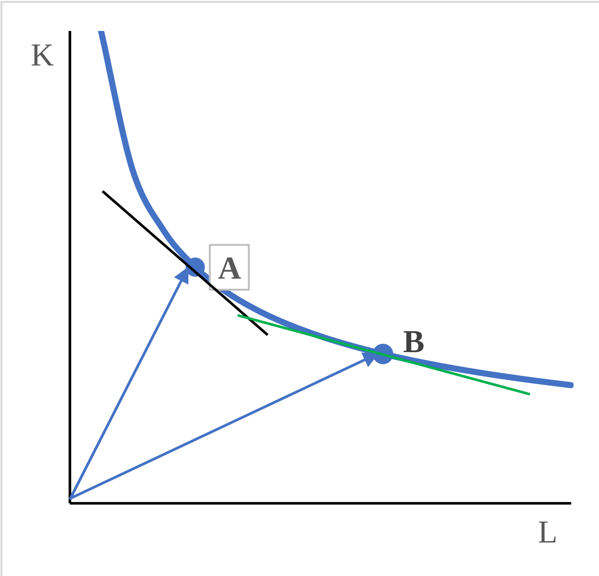

Fig. 44. Changes in factor intensity and changes in MRS

Given the relationship of elasticity with the derivative

$$\sigma = \frac{d(K/L)}{dMRS} \cdot \frac{MRS}{K/L},$$

here $MRS = -\frac{dK}{dL}$.

The elasticity of substitution shows by how many percent it is necessary to change the factor intensity (capital-labor ratio K/L) in order to achieve a 1% change in the marginal rate of substitution (capital-labor increments ratio $|\Delta K|/|\Delta L|$).

The elasticity of substitution is a pure number independent of the units of measurement of K and L, since both the numerator and the denominator in the elasticity formula are measured in the same units.

The capitalx-labor ratio is the slope of the radius vector drawn from the origin to a point on the isoquant. The rate of technical substitution shows the slope of the isoquant. Fig. 16 shows how the capital-labor ratio and the rate of technical substitution change as we move along the isoquant. In fig. 44 process A is more capital intensive than process B. The upper part of the isoquant includes more capital-intensive processes. The lower part of the isoquant includes more labor-intensive techniques.

The elasticity of substitution for different production functions is shown in the tabl. 8:

Tabl. 8. The elasticity of substitution

Production function	**Elasticity of substitution**	
Cobb–Douglas production function $Y = a_0 K^{a_1} L^{a_2}$	$\sigma = 1$	constant substitutability equal to 1
Linear production function $Y = AK + BL$	$\sigma = \infty$	perfect substitutability
Leontief production function $Y = min(aK, bL)$	$\sigma = 0$	zero substitutability

4.3. The Cobb–Douglas Production Function

The **Cobb–Douglas production function** is widely used to represent the technological relationship between the amounts of two inputs (physical capital and labor) and the amount of output that can be produced by those inputs. The Cobb–Douglas production function was developed and tested against statistical evidence by Charles Cobb and Paul Douglas between 1927–1947.

The Cobb–Douglas production function has the form

$$Y = a_0 K^{a_1} L^{a_2}, \tag{4.1}$$

where Y – total production (the real value of all goods produced in a year), K – capital input (a measure of all machinery, equipment, and buildings; the value of capital input divided by the price of capital), L – labor input (person-hours worked in a year), a_0, a_1, a_2 – function parameters are positive constants, and $0 < a_1 < 1$, $0 < a_2 < 1$.

Although each of the coefficients a_1 and a_2, is less than 1, their sum may be less than, equal to, or greater than 1. This sum shows **the effect of a simultaneous proportional increase in inputs**, both the amount of capital and the amount of labor.

Let increase the amounts of inputs by a factor of m. So, let the initial value of the output:

$$Y_0 = a_0 K^{a_1} L^{a_2}.$$

The new value of the output as a result of increasing the amount of inputs by a factor of *m* will be:

$$Y_1 = a_0 (mK)^{a_1} (mL)^{a_2} = m^{a_1 + a_2} a_0 K^{a_1} L^{a_2} = m^{a_1 + a_2} Y_0 \,.$$

If $a_1 + a_2 = 1$, then an increase in inputs by a factor of *m* leads to an increase in output by a factor of *m* as well. Function displays **constant returns to scale**, which means that doubling the use of capital *K* and labor *L* will also double output *Y*.

If $a_1 + a_2 < 1$, then output will increase less than *m* times, **returns to scale are decreasing**, and if $a_1 + a_2 > 1$, then output will increase by more than *m* times, **returns to scale are increasing**.

The analysis of the production function uses average and marginal values. Let's calculate the average and marginal values for the Cobb–Douglas production function:

The average product of a variable factor is the output per unit of variable factor. **The average product of labor** is the ratio of output to the number of units of labor:

$$AP_L = \frac{Y}{L} \,.$$

Dividing both sides of equation (1) by *L*, we obtain the average product of labor

$$AP_L = \frac{Y}{L} = \frac{a_0 K^{a_1} L^{a_2}}{L} = a_0 K^{a_1} L^{a_2 - 1}.$$

Since the coefficient $0 < a_2 < 1$, the degree indicator $a_2 - 1$ is a negative value. Consequently, as the amount of labor increases, the average product of labor decreases.

Similarly, we can calculate the average product of another factor – capital. We get the average value, which is called **the average product of capital**

$$AP_K = \frac{Y}{K} = \frac{a_0 K^{a_1} L^{a_2}}{K} = a_0 K^{a_1 - 1} L^{a_2}.$$

Let calculate the marginal products. The partial derivatives of the production function are called marginal products and show the change in output per unit change in the amount of a variable factor. **Marginal product of labor** is the partial derivative of the production function with respect to the variable L

$$MP_L = \frac{\partial Y}{\partial L} = \lim_{\Delta L \to 0} \frac{\Delta Y}{\Delta L} \approx \frac{\Delta Y}{\Delta L}$$

and **marginal product of capital** is the partial derivative of the production function with respect to the variable K

$$MP_K = \frac{\partial Y}{\partial K} = \lim_{\Delta K \to 0} \frac{\Delta Y}{\Delta K} \approx \frac{\Delta Y}{\Delta K}.$$

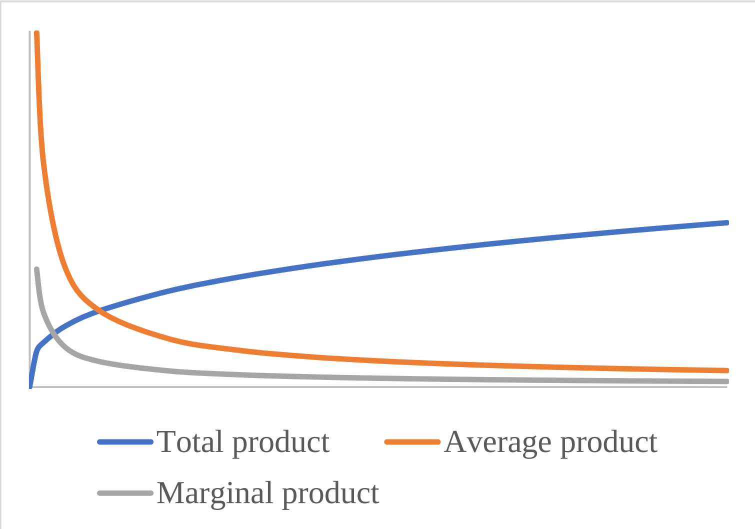

Fig. 45. Dependencies of total, average and marginal product on the amount of inputs

Let's take partial derivatives of the Cobb–Douglas production function $Y = a_0 K^{a_1} L^{a_2}$

$$\frac{\partial Y}{\partial K} = a_0 a_1 K^{a_1 - 1} L^{a_2} = \frac{a_1 Y}{K}, \qquad (4.2)$$

$$\frac{\partial Y}{\partial L} = a_0 a_2 K^{a_1} L^{a_2 - 1} = \frac{a_2 Y}{L}. \qquad (4.3)$$

Here $Y/K = AP_K$ is average product of capital; $Y/L = AP_L$ is average product of labor. So, the marginal products are proportional to the average products, a_1, a_2 are coefficients of proportionalities:

$$\frac{\partial Y}{\partial K} = a_1 AP_K, \quad \frac{\partial Y}{\partial L} = a_2 AP_L. \qquad (4.4)$$

Since $a_1 < 1$, $a_2 < 1$, then we can conclude from formulas (4.4) that the marginal products are less than the average products (fig. 45).

The average and marginal products decrease as the amount of the factor increases in the case of a Cobb–Douglas production function (fig. 45).

Elasticities of output with respect to inputs are calculated to study the dependence of output on inputs. Elasticity of output with respect to capital is

$$E_K(Y) = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y},$$

elasticity of output with respect to labor

$$E_L(Y) = \frac{\partial Y}{\partial L} \cdot \frac{L}{Y}.$$

Using the formulas (4.2)–(4.3), let's calculate the output elasticities in the case of the Cobb–Douglas production function

$$E_K(Y) = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = a_1 \frac{Y}{K} \cdot \frac{K}{Y} = a_1,$$

$$E_L(Y) = \frac{\partial Y}{\partial L} \cdot \frac{L}{Y} = a_2 \frac{Y}{L} \cdot \frac{L}{Y} = a_2.$$

Thus, the degree indicators of the Cobb–Douglas function a_1 and a_2 are the output elasticity of capital and labor, respectively. These elasticities are constants determined by available technology.

Output elasticity is the percentage change of output divided by the percentage change of an input. Output elasticity measures the responsiveness of output to a change in levels of either labor or capital used in production. Output elasticities do not depend on the amount of inputs for the Cobb–Douglas production function. A 1% increase in capital inputs leads to approximately a a_1% increase in output; and a 1% increase in labor inputs causes approximately a a_2% increase in output for all combinations of inputs.

The parameter a_0 is a total factor productivity. The greater the value of the parameter a_0, the greater the output when the coefficients a_1, a_2 and inputs K, L remain unchanged.

Using formulas (4.2)–(4.3), we find the marginal rate of substitution of labor for capital for the Cobb–Douglas production function:

$$MRS = \frac{\partial Y / \partial L}{\partial Y / \partial K} = \frac{a_2 \, Y/L}{a_1 \, Y/K} = \frac{a_2}{a_1} \frac{K}{L} = \frac{a_2}{a_1} k,$$

where $k = K/L$ is the factor intensity (capital-labor ratio).

Thus, the marginal rate of substitution of labor for capital is proportional to the factor intensity for the Cobb–Douglas production function. The elasticity of substitution is equal to 1:

$$\sigma = \frac{dk}{dMRS} \cdot \frac{MRS}{k} = \frac{a_1}{a_2} \cdot \frac{a_2}{a_1} = 1 .$$

Example 1. Let some production be represented by a Cobb–Douglas production function. One employee produces products equal to 1 million rubles per month. The total number of employees is 1000 people. The capital used is estimated at 10 billion rubles. It is known that in order to increase output by 3% it is necessary to increase either the amount of capital by 6% or the number of employees by 9%.

1. Make a production function for this enterprise, calculating the output elasticities.

2. Calculate the average and marginal products of labor and capital.

3. Calculate the marginal rate of substitution of labor for capital.

Solving: The Cobb–Douglas production function has the following form:

$$Y = a_0 K^{a_1} L^{a_2} .$$

1. Let's find the output elasticities a_1 and a_2. It is known that output increases by 3% when the amount of capital increases by 6% or the number of employees increases by 9%. Then the elasticity of output with respect to capital

$$a_1 = E_K(Y) \approx \frac{\%\Delta Y}{\%\Delta K} = \frac{3\%}{6\%} = \frac{1}{2},$$

and the elasticity of output with respect to labor

$$a_2 = E_L(Y) \approx \frac{\%\Delta Y}{\%\Delta L} = \frac{3\%}{9\%} = \frac{1}{3}.$$

So, the production function has the form:

$$Y = a_0 K^{1/2} L^{1/3}.$$

We find the parameter a_0 from the condition that when the amounts of labor $L = 1000$ and capital used $K = 10^{10}$ output is $Y = 10^6 \cdot L = 10^9$:

$$a_0 = \frac{Y}{K^{1/2} L^{1/3}} = \frac{10^9}{\sqrt{10^{10}} \cdot \sqrt[3]{1000}} = \frac{10^9}{10^5 \cdot 10} = 10^3 = 1000.$$

So, the production function has the form: $Y = 1000K^{1/2}L^{1/3}$, $a_1 + a_2 = 1/2 + 1/3 = 5/6 < 1$, therefore, returns to scale are decreasing.

2. Let's calculate the average and marginal products of labor:

$$AP_L = \frac{Y}{L} = \frac{10^9}{1000} = 10^6\ ,$$

$$MP_L = a_2 \cdot AP_L = \frac{1}{3} \cdot 10^6 = 333333\frac{1}{3}.$$

Thus, output per employee is 1 million rubles per month. The increase in output is 333333.33 rubles when the number of employees increases by 1 person.

Let's calculate the average and marginal products of capital:

$$AP_K = \frac{Y}{K} = \frac{10^9}{10^{10}} = 0.1\ ,$$

$$MP_K = a_1 \cdot AP_K = \frac{1}{2} \cdot 0.1 = 0.05.$$

The output per ruble of capital used is 0.1 ruble per month. The increase in output is 0,05 rubles when the amount of capital increases by 1 ruble.

3. The marginal rate of substitution of labor for capital is

$$MRS = \frac{MP_L}{MP_K} = \frac{333333\frac{1}{3}}{0.05} = 6666666\frac{2}{3}.$$

It is necessary to increase the use of capital by 666666.67 rubles in order to reduce one employee and keep the monthly output equal to 1 billion rubles.

Example 2. The production function $Y = 3.2K^{0.6}L^{0.4}$ is given, where *Y* is the output in monetary terms, *K* is the cost of capital assets, and *L* is the wage fund. The factors of production have changed: the wage fund decreased by 3%, the cost of capital assets increased by 2%. By how many percent will output, average product of labor and average product of capital change?

Solving:

1) Take the logarithm of the production function

$$ln\,Y = ln(a_0K^{a_1}L^{a_2}) = ln(a_0) + a_1\,ln(K) + a_2\,ln(L).$$

Let's take the derivative of this equality

$$\frac{dY}{Y} = a_1 \frac{dK}{K} + a_2 \frac{dL}{L}.$$

This equality can be written in finite increments:

$$\frac{\Delta Y}{Y} \approx a_1 \frac{\Delta K}{K} + a_2 \frac{\Delta L}{L},$$

Where $\Delta K/K$, $\Delta L/L$ are the relative increments in capital assets and wage fund, respectively. In this example they are 0.02 and -0.03. Then the change in output

$$\frac{\Delta Y}{Y} \approx 0.6 \cdot 0.02 + 0.4 \cdot (-0.03) = 0.012 - 0.012 = 0.$$

Therefore, the output does not change.

2) Calculate the average product of labor

$$AP_L = \frac{Y}{L} = \frac{3.2K^{0.6}L^{0.4}}{L} = 3.2K^{0.6}L^{-0.6}.$$

Taking the logarithm of this equality, we obtain

$$ln(AP_L) = ln(3.2K^{0.6}L^{-0.6}) = ln(3.2) + 0.6\,ln(K) - 0.6\,ln(L).$$

Let's take the derivative of this equality and write down the result using the finite increments

$$\frac{\Delta AP_L}{AP_L} \approx 0.6\frac{\Delta K}{K} - 0.6\frac{\Delta L}{L} = 0.6 \cdot 0.02 - 0.6 \cdot (-0.03) = 0.6 \cdot 0.05 = 0.03.$$

Thus, the average product of labor increased by 3%.

3) Let's find the change in the average product of capital

$$AP_K = \frac{Y}{K} = \frac{3.2K^{0.6}L^{0.4}}{K} = 3.2K^{-0.4}L^{0.4}.$$

Take the logarithm of this equality

$$ln(AP_K) = ln(3.2K^{-0.4}L^{0.4}) = ln(3.2) - 0.4\,ln(K) + 0.4\,ln(L).$$

Taking the derivative of this equality and using finite increments, we obtain

$$\frac{\Delta AP_K}{AP_K} \approx -0.4\frac{\Delta K}{K} + 0.4\frac{\Delta L}{L} = -0.4 \cdot 0.02 + 0.4 \cdot (-0.03) = -0.4 \cdot 0.05 = -0.02.$$

So, the average product of capital decreased by 2%.

Example 3. Find the marginal rate of substitution of labor for capital if the production function is $Y = 1000K^{1/2}L^{1/3}$, the total number of employees is 1000 people, the used capital is 10 billion rubles.

Solving: The factors of production are $K = 10^{10}, L = 1000$. Let's calculate factor intensity (capital-labor ratio).

$$k = \frac{K}{L} = \frac{10^{10}}{10^3} = 10^7.$$

This means that there are 10 million rubles of capital assets per 1 employee. Then the marginal rate of substitution of labor for capital is

$$MRS = \frac{a_2}{a_1}k = \frac{1/3}{1/2} \cdot 10^7 = \frac{2}{3} \cdot 10^7 = 6666666\frac{2}{3} \ .$$

Therefore, 6666666.67 rubles must be invested in capital assets in order to reduce one employee.

REVIEW QUESTIONS

Section 1

1. What is a functional dependence? Give examples of functional dependencies in economics.

2. What is a positive relationship? What is a negative relationship?

3. How are absolute, relative, and percentage changes in the variable *x* calculated?

4. What is a measure of the change in an economic indicator under the influence of a change in its determinant?

5. How is functional dependence used as a decision-making tool? Give examples.

6. What is the elasticity of a function? What does it show?

7. What are the properties of elasticity?

8. What is the elasticity of a linear function? What is the elasticity of a power function? What is the elasticity of an exponential function?

Section 2

1. Give an example of a linear demand function that shows the dependence of quantity demanded on price. Plot a graph of this function.

2. Give an example of a linear supply function that shows the dependence of supply on price. Plot a graph of this function.

3. What is the equilibrium price? What is the equilibrium sales volume?

4. What are price elastic demand and price inelastic demand?

5. How does a change in price influence the change in total revenue when demand is price elastic and when demand is price inelastic?

6. Give an example of a linear demand function that shows the dependence of demand on income. Plot a graph of this function.

7. How do we know from the income elasticity of demand whether a good is normal or inferior?

8. How do we know from cross-elasticity of demand whether goods are substitutes or complementary?

Section 3

1. What is a utility function?

2. What is the indifference curve?

3. One good costs 10 rubles, and another good costs 25 rubles. The consumer spends 100 rubles to buy these goods. What is equation of his budget line? Draw budget line. Which point at the budget line is preferable?

4. Diego has utility function $u = 3x_1 + 5x_2$, where x_1 is number of apples and x_2 is number of cookies. What are his marginal utility for apples and marginal utility for cookies?

5. Maria's utility function is $u = x_1 + 2x_2$. Describe the location of her optimal bundle in term of relative prices of goods.

6. Nydia likes chicken and pizza. Her utility function is $u = 10x_1^2 x_2$. Her weekly income is $90, which she spends on only chicken and pizza. She pays $10 for chicken and $5 for pizza. Show her budget line, indifference curve and optimal bundle on your graph.

7. What is substitution effect?

8. What is income effect?

Section 4

1. What is a production function?

2. What is the average product of labor and the average product of capital? Find them for the Cobb–Douglas production function.

3. What is the marginal product of labor and the marginal product of capital? Find them for Leontief production function.

4. What is the relationship between average product and marginal product in the case of the Cobb–Douglas production function?

5. What are the elasticities of output with respect to inputs in the case of the Cobb–Douglas production function?

6. What is an isoquant? Draw isoquants of the linear function, the Cobb–Douglas function and the Leontief function.

7. What is a measure of the substitutability of factors of production?

8. Draw isoquants in the case of perfect substitutability of production factors and in the case of perfect complementary factors.

TEST

1. Select correct statements: The elasticity of a function $y = f(x)$ is equal to

A. the derivative of a function multiplied by x and divided by y.

B. the limit of the ratio of the absolute change in variable y to the absolute change in variable x.

C. the limit of the ratio of the relative change in variable y to the relative change in variable x.

D. the limit of the ratio of the percentage change in variable y to the percentage change in variable x.

2. The elasticity of a function $y = f(x)$ can be calculated using the formula

A. $E_x(y) = y'\frac{y}{x}$ B. $E_x(y) = xyy'$ C. $E_x(y) = y'\frac{x}{y}$ D. $E_x(y) = \frac{y'}{xy}$

3. Demand is price elastic if

A. $|E_p(q)| < 1$ B. $|E_p(q)| = 1$ C. $E_p(q) = 0$ D. $|E_p(q)| > 1$

4. When demand is price inelastic an increase in price leads to

A. decrease in total revenue

B. increase in total revenue

C. zero total revenue

D. negative total revenue

5. The income elasticity of demand is positive. This means

A. demand increases as the price increases

B. demand is not dependent on income

C. the good is inferior

D. the good is normal

6. The elasticity of the power function $y = x^\alpha$ is

A. $\alpha x^{\alpha-1}$ B. α C. $1-\alpha$ D. $1+\alpha$

7. If $y = u \cdot v$ then elasticity $E_x(y)$ is equal to

A. $E_x(u) - E_x(v)$ B. $E_x(u) + E_x(v)$ C. $\frac{E_x(u)}{E_x(v)}$ D. $E_x(u) \cdot E_x(v)$

8. A 5% increase in the price of the good caused a 10% decrease in demand. What is price elasticity of demand?

A. -2 B. -0.5 C. 0.5 D. 2

9. What is elasticity of the function $y = \frac{x^3+2}{x+8}$ at the point $= 2$?

A. 1.8 B. 2 C. 2.2 D. 2.6

10. The demand function is $q = 247 - 6p$, and the supply function is $s = p^2$, where p is the price of the good. What is the equilibrium price?

A. 20 B. 17 C. 15 D. 13

11. The demand function is $q = \frac{p+10}{p+1}$, and the supply function is $s = 2p + 3.5$, where p is the price of the good. What is the equilibrium sales volume?

A. 5.5 B. 16.5 C. 1 D. 10

12. What is the measure of consumer satisfaction with a consumption bundle, which is the higher the more preferable the bundle?

A. demand function

B. utility function

C. marginal utility

D. supply function

13. What is the level line of the utility function?

A. isoquant

B. income-consumption curve

C. budget line

D. indifference curve

14. The utility function is $u = 6\sqrt{x_1} + x_2$. What is the indifference curve?

A. $6\sqrt{x_1} + x_2 = C$ B. $\frac{3}{\sqrt{x_1}} + 1 = C$ C. $\frac{6\sqrt{x_1}}{x_2} = C$ D. $6\sqrt{x_1}x_2 = C$

15. The consumer has a utility function $u = x_1 x_2$. Good 1 costs 20 rubles, good 2 costs 10 rubles. The consumer spends 200 rubles to buy these goods. What is the optimal consumption bundle?

A. $x_1 = 0, x_2 = 20$ B. $x_1 = 8, x_2 = 4$ C. $x_1 = 5, x_2 = 10$ D. $x_1 = 10, x_2 = 10$

16. What is the optimal consumption bundle?

A. point of tangency of the indifference curve with the budget line

B. point of intersection of the indifference curve with the budget line

C. point of tangency of the isoquant with the isocost

D. point of tangency of the indifference curve with the isoquant

17. When the utility function is $u(x_1, x_2) = x_1 x_2$ an increase in the price of one of the goods by a factor of *k* leads to

A. a decrease in demand for this good by a factor of *k*

B. a decrease in demand for this good by a factor of $\sqrt{k}$

C. a increase in consumer costs by a factor of $\sqrt{k}$

D. increase in demand for another good by a factor of *k*

18. The production function is $Y = 2K^{0.6}L^{0.51}$. What is the elasticity of output with respect to capital?

A. 1.11 B. 0.6 C. 0.51 D. 3.11

19. The production function has the form $Y = K^{0.6}L^{0.3}$, where *K* is invested capital, *L* is labor input. Then an increase in invested capital by 1% leads to an increase in output by

A. 0.6% B. 0.3% C. 0.9% D. 0.5%

20. The production function is $Y = K^{0.5}L^{0.5}$, where *K* is capital input, *L* is labor input. What is the marginal product of labor $\frac{\partial Y}{\partial L}$ if $K = 4$, $L = 25$?

A. 0.4 B. 0.2 C. 1.25 D. 2.5

Answers: 1. A, C, D; 2. C; 3. D; 4. B; 5. D; 6. B; 7. D; 8. A; 9. C; 10. D; 11. A; 12. B; 13. D; 14. A; 15. C; 16. A; 17. A; 18. B; 19. A; 20. B.

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